Notes on Factorization Algebras and Lie Algebra Representations

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Abstract

These are some notes prepared for a series of talks during Summer 2022 at RIMS. We explain in more detail the main techniques used in [CF21].

Contents

1 Talk 1: Factorization Structure

Readers are recommended to refer to either the talk slides or the introductory section of [CF21] for the overall strategy of this proof.

Categorical Background The method we use here crucially relies on higher category theory and homotopical algebra, and one would be unwise to try to summarize those in a few talks. We take the approach of compelling our readers to believe that 1-category theory and (colored) operad theory have all been ported to the ∞-world, and there is enough tools in the box that we can effectively manipulate them as we do in the classical setting. We promise that we will point out whenever homotopy coherence is a serious issue and requires careful handling. Our basic terminologies can also be found in the first chapter of [GR17].

A few examples of this blackboxing: for us, "category" will always mean an $(\infty, 1)$-category in the sense of [Lur09]. A "commutative" object means an algebra over the topological $\mathbb{E}_\infty$-operad, an "associative" object analogously for the $\mathbb{E}_1$-operad.

1.1 Ran Space and Non-unital Factorization

We start by giving a precise definition to factorization structure using the Ran space.

Definition 1.1.1. Let $\mathsf{Comm}_{\mathrm{nu}}$ denote the category of (non-empty) finite sets and surjective maps between them. For any prestack (for us those will always be ∞-groupoid valued) X, we define $\mathrm{Ran}(X)$ to be the colimit

$$\mathrm{Ran}(X) := \mathrm{colim}_{I \in (\mathsf{Comm}_{\mathrm{nu}})^{\mathrm{op}}} X^I$$

in the category of prestacks.

Note that this indexing set is *not* filtered, so the result is usually only a prestack (and not have any descent property) even when X is a scheme. When X is set-valued, one can prove[1] that $\mathrm{Ran}(X)(S)$ identifies with the set of non-empty subsets of $X(S)$.

Let Shv be any "large" sheaf theory (e.g. D-modules, ind-constructible sheaves, ind-holonomic D-modules); "large" here means $\mathrm{Shv}(X)$ is cocomplete for any X (so e.g. the category of *holonomic* D-modules is not large). For $X \to Y$, assume we have a !-pullback functor $\mathrm{Shv}(Y) \to \mathrm{Shv}(X)$. Then we define

$$\mathrm{Shv}(\mathrm{Ran}(X)) := \lim_{I \in (\mathsf{Comm}_{\mathrm{nu}})^{\mathrm{op}}} \mathrm{Shv}(X^I).$$

Let us see what such an object is for lowest strata. So suppose $\mathcal{F}$ is a sheaf on $\mathrm{Ran}(X)$.

- Its restriction to X (the main diagonal, i.e. $|I| = 1$) gives a sheaf $\mathcal{F}|_X \in \mathrm{Shv}(X)$;
- Its restriction to X^2 gives a sheaf $\mathcal{F}|_{X^2} \in \mathrm{Shv}(X^2)$;
- The surjective map $p : 2 \twoheadrightarrow 1$ imposes the condition that $\Delta^!(\mathcal{F}|_{X^2}) \simeq \mathcal{F}|_X$;
- The non-identity map $s : 2 \twoheadrightarrow 2$ imposes the condition that $\mathcal{F}|_{X^2}$ is S^2-equivariant;
- That $p \simeq p \circ s$ implies that the S^2-equivariance must be identity upon restricting to the diagonal.

Unstable variants There are two variants we will consider. First is by replacing Shv above with the following assignment:

[1] The π_0 identification is easy, and one shows higher π_i vanish using the "contractibility trick".

Definition 1.1.2. The functor ShvCat maps each scheme X to the (large) category

$$\mathrm{QCoh}(X)\text{-mod}(\mathrm{DGCat}),$$

where $\mathrm{QCoh}(X)$ is given a symmetric monoidal structure via the $*$-tensor product. For $X \to Y$ the corresponding map $\mathrm{QCoh}(Y)\text{-mod}(\mathrm{DGCat}) \to \mathrm{QCoh}(X)\text{-mod}(\mathrm{DGCat})$ is induced by the $*$-pushforward functor $\mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$.

The second one is given by the functor

$$\mathrm{PreStk}_{/\bullet} : X \mapsto \mathrm{PreStk}_{/X} \in \mathrm{Cat},$$

where the functoriality is given by fiber product. Then an element $\mathrm{PreStk}_{/\bullet}(\mathrm{Ran})$ is called a sheaf of spaces over $\mathrm{Ran}(X)$.

Remark 1.1.3. The crucial difference here is that, whereas every $\mathrm{Shv}(X)$ that we'll consider admits Cousin triangles for open-closed embeddings, this is very false for the two cases above.

Variant: Crystals Recall the de Rham prestack $\mathcal{Y}_{\mathrm{dR}}$ associated with any prestack $\mathcal{Y}$, whose $\mathrm{Spec}(A)$ points is given by $\mathcal{Y}(\mathrm{Spec}(A^{\text{classical, reduced}}))$. The main property of this object is such that for reasonable (locally almost of finite type) prestacks X, one has $\mathrm{DMod}(X) \simeq \mathrm{QCoh}(X_{\mathrm{dR}})$. We have $\mathrm{Ran}(X)_{\mathrm{dR}} \simeq \mathrm{Ran}(X_{\mathrm{dR}}) \simeq \mathrm{colim}_I\, X_{\mathrm{dR}}^I$, and the notation

$$\mathrm{ShvCat}(\mathrm{Ran}(X)_{\mathrm{dR}}) \quad \mathrm{PreStk}_{/\bullet}(\mathrm{Ran}_{\mathrm{dR}})$$

can be made sense of just as before. These objects will be called *crystals of categories/spaces* over $\mathrm{Ran}(X)$.

Example 1.1.4. Given a crystal of categories $\mathcal{F}$ over $\mathrm{Ran}(X)$, its base change to the main diagonal X_{dR} is a module category over the symmetric monoidal category $\mathrm{DMod}(X)$.

Correspondence Structure The additional structure of the non-unital Ran space is that it is a commutative algebra object in the category of *correspondences* of prestacks, via the following diagram (as well as higher levels)

$$\mathrm{Ran}(X) \times \mathrm{Ran}(X) \xleftarrow{\text{inclusion}} [\mathrm{Ran}(X) \times \mathrm{Ran}(X)]_{\mathrm{disj}} \xrightarrow{\text{union}} \mathrm{Ran}(X)$$

where $[\mathrm{Ran}(X) \times \mathrm{Ran}(X)]_{\mathrm{disj}}$ is a certain prestack whose $\mathbb{C}$-points, let's say when X is set-valued, consists of pairs of $\mathbb{C}$-points of $\mathrm{Ran}(X)$ that are disjoint.

Remark 1.1.5. The correct definition of S-valued points of this prestack is a bit tricky and is not really useful for our discussion.

There is a Grothendieck fibration over the correspondence categories of prestacks

$$\mathcal{SHV} \to \mathrm{PreStk}_{\mathrm{corr}}$$

whose objects are pairs $(X \in \mathrm{PreStk}, \mathcal{F} \in \mathrm{Shv}(X))$, and whose morphisms $(X, \mathcal{F}) \to (Y, \mathcal{G})$ consists of a correspondence

$$X \xleftarrow{\alpha} Z \xrightarrow{\beta} Y$$

and a map $\gamma : \alpha^!(\mathcal{F}) \to \beta^!(\mathcal{G})$ in $\mathrm{Shv}(Z)$. It admits a symmetric monoidal structure via box product.

Definition 1.1.6. A non-unital weak factorization algebra is a commutative algebra object in $\mathcal{SHV}$ that maps to $\mathrm{Ran}(X)$ via the forgetful functor. A non-unital factorization algebra is one where γ as above is an isomorphism.

Similarly, the notion of a non-unital factorization *category* and *space* can also be made sense of.

Example 1.1.7. Unfolding definition, we see that being a weak factorization is the following additional data: let $j : X^2 \smallsetminus X \to X^2$ be the inclusion. Then a weak factorization algebra in particular entails a morphism

$$j^*(\mathcal{F}|_X \boxtimes \mathcal{F}|_X) \to j^*(\mathcal{F}|_{X^2})$$

which is required to be an isomorphism for factorization algebras.

Remark 1.1.8. The category of non-unital factorization algebras is equivalent to the category if *chiral algebras* defined by Beilinson and Drinfeld via the *chiral Koszul duality.* We'll explain this in lecture if needs be.

Example 1.1.9. Recall the BD affine Grassmannian $\mathrm{Gr}_{G,\mathrm{Ran}_{\mathrm{dR}}}$ and note that it is a non-unital factorization crystal of spaces in the sense above; we denote its base change back to Ran by $\mathrm{Gr}_{G,\mathrm{Ran}}$, which is a non-unital factorization space. Let us now observe that $\mathrm{DMod}(\mathrm{Gr}_{G,\mathrm{Ran}})$ is a non-unital factorization category. It suffices to check that for every surjection $I \twoheadrightarrow J$ we have

$$\mathrm{DMod}(\mathrm{Gr}_{G,X^J}) \simeq \mathrm{DMod}(\mathrm{Gr}_{G,X^I}) \otimes_{\mathrm{DMod}(X^I)} \mathrm{DMod}(X^J)$$

which follows from the fact that $\mathrm{Gr}_{G,X^J} \simeq \mathrm{Gr}_{G,X^I} \times_{X^I} X^J$ and the general base-change property of the category of D-modules. (Note that such properties do not hold for ind-coherent sheaves.)

1.2 Unital Structure

The following is a fun object:

Definition 1.2.1. The unital Ran space $\mathrm{Ran}_{\mathrm{un}}(X)$ is the partially oplax colimit of the diagram $I \mapsto X^I$ ranging over $I \in \mathsf{Comm}^{\mathrm{op}}$ (the set of *all*—potentially empty—finite sets and *all* maps between them) which is strict over the subdiagram ${\mathsf{Comm}_{\mathrm{nu}}}^{\mathrm{op}}$. Here the colimit is taken in the category of *lax* prestacks (i.e. Cat-valued prestacks).

Here's what this actually means. When $X(S)$ is a set, $\mathrm{Ran}_{\mathrm{un}}(X)(S)$ is the category of all (potentially empty) subsets of $X(S)$, where the arrows are inclusion of subsets.

Note that Shv is itself a lax prestack. For any lax prestack $\mathcal{Y}$, we define $\mathrm{Shv}(\mathcal{Y})$ to be the category of morphisms $\mathcal{Y} \to \mathrm{Shv}$ of lax prestacks. In the case of $\mathrm{Ran}_{\mathrm{un}}(X)$, it is not difficult to show that $\mathrm{Shv}(\mathrm{Ran}_{\mathrm{un}}(X))$ can be described as a partially lax limit.

Example 1.2.2. Let's again see what this data actually is in lowest strata. Suppose $\mathcal{F}$ is a sheaf over the unital Ran space, then, in addition to the non-unital data, we have:

- The empty set $I = \varnothing$ gives an element $\mathcal{F}_0 := \mathcal{F}|_{X^\varnothing} \in \mathrm{DMod}(\mathrm{pt}) \simeq \mathrm{Vect}$. It is usually useful to normalize by requiring this element to be $\mathbb{C}$;
- The inclusion $\varnothing \to k$ for each k gives a *map* $u_k : \mathcal{F}_0 \otimes \omega_{X^k} \to \mathcal{F}|_{X^k}$ (not required to be an isomorphism). If we use the normalization $\mathcal{F}_0 = \mathbb{C}$, then this is a section of each $\mathcal{F}|_{X^k}$ called the unit section;

- The two inclusions $\iota_l, \iota_r : 1 \to 2$ gives two maps $v_l : \mathcal{F}|_X \boxtimes (\mathcal{F}_0 \otimes \omega_X) \to \mathcal{F}|_{X^2}$ and $v_r : (\mathcal{F}_0 \otimes \omega_X) \boxtimes \mathcal{F}|_X \to \mathcal{F}|_{X^2}$;
- The fact that $(\varnothing \to 2 \xrightarrow{p} 1) \simeq (\varnothing \to 1), (s \circ \iota_l) \simeq (\iota_r), (p \circ \iota_l) \simeq (p \circ \iota_r) \simeq \mathrm{id}$ imposes further compatibility relations for the data above.

Using the unital Ran space everywhere in the discussion above, we obtain the notion of *unital* factorization algebras / spaces / categories.

Remark 1.2.3. It is a theorem of Lurie that for non-unital $\mathbb{E}_2$-algebras, being unital is a *property* (that is, the category of unital $\mathbb{E}_2$-algebras embeds fully faithfully into the category of non-unital ones). It is unclear to me whether this is true for factorization algebras, though it is clear that it is very false for factorization *categories*.

Example 1.2.4. $\mathrm{Gr}_{G,\mathrm{Ran}}$ is a unital factorization space. The additional data is the following: for each $s \in X^I(S), t \in X^J(S)$ such that $\Gamma_s \subseteq \Gamma_t$ (here recall that $\Gamma_s \subseteq S \times X$ is the set-theoretic union of the graphs of maps in s), we have a map

$$S \times_{X^I} \mathrm{Gr}_{G,X^I} \to S \times_{X^J} \mathrm{Gr}_{G,X^J}$$

given by restricting the trivialization on $S \times X - \Gamma_s$ to $S \times X - \Gamma_t$.

Example 1.2.5. Here we encounter a tricky point: if we follow the discussion above, we see that the loop space itself $G(K)_{\mathrm{Ran}}$ has no unital structure! This is a feature, not a bug.

Definition 1.2.6. Consider the lax prestack $\mathrm{PreStk}^{\mathrm{corr}}_{/\bullet}$ where the underlying groupoid of $\mathrm{PreStk}^{\mathrm{corr}}_{/\bullet}(S)$ for S affine is still $\mathrm{PreStk}_{/S}$, but the morphisms are *correspondences* of prestacks over S. A corr-sheaf of spaces over $\mathrm{Ran}_{\mathrm{un}}$ is then a map $\mathrm{Ran}_{\mathrm{un}} \to \mathrm{PreStk}^{\mathrm{corr}}_{/\bullet}$ of lax prestacks.

Using $X \mapsto \mathrm{CorrSheafSpc}_{/X}$ in lieu of $X \mapsto \mathrm{PreStk}_{/X}$ one defines the notion of a corr-unital factorization space. (This is too flexible a notion; we usually require that the underlying non-unital object is an actual factorization space, i.e. the corr-ness only happens for the unital maps.)

More explicitly, if we expand the definition as in Example 1.2.4, then for a corr-unital factorization space $\{\mathcal{Y}\}_{X^I}$, we only require a *correspondence* over S:

$$S \times_{X^I} \mathcal{Y}_{X^I} \leftarrow \mathcal{Y}_{I,J} \to S \times_{X^J} \mathcal{Y}_{X^J}.$$

Variant 1.2.7. If we flip the arrow in Example 1.2.4, i.e. demand a map

$$S \times_{X^J} \mathrm{Y}_{X^J} \to S \times_{X^I} \mathrm{Y}_{X^I}$$

instead, we obtain the notion of a *co-unital* factorization space. Clearly corr-unital factorization space is a common generalization for both unital and co-unital cases.

Example 1.2.8. $G(O)_{X^I}$ is a co-unital factorization space (by unfolding definition), and $G(K)_{X^I}$ is a corr-unital factorization space, where $G(K)_{I,J}$ classifies the following data:

- A J-tuple of maps $x_j : S \to X$;
- A map $D_{\Gamma_t} \smallsetminus \Gamma_s \to G$.

Warning 1.2.9. The laxness of our notions means there are many different notions of *morphisms* between corr-unital factorization spaces, which one has to keep track of. This is getting too technical, so we refer readers to [CF21, Variant 10.3.12] for details.

Example 1.2.10. The category $\mathrm{DMod}(G(K)_{\mathrm{Ran}})$ admits a unital factorization category structure via *pull-push* along the corr-unital correspondence diagrams. This in particular means we need to handle base change, so homotopy coherence must be treated carefully. The general machinery of category of correspondences in Gaitsgory-Rozenblyum can be utilized here.

Given a (unital or non-unital) factorization category, one can make sense of factorization algebras internal to it, by using the functor

$$\mathrm{ShvCatSect} : X \mapsto \{(C \in \mathrm{QCoh}(X)\text{-mod}(\mathrm{DGCat}), A \in C)\}$$

we leave details to the reader.

Upon unfolding definition, one sees that the unit sections for a unital factorization category is automatically a unital factorization algebra inernal to it, which we'll call the *vacuum* of the factorization category.

Example 1.2.11. The vacuum of $\mathrm{DMod}(\mathrm{Gr}_{G,\mathrm{Ran}})$ is the δ-sheaf at the unit section; the vacuum of $\mathrm{DMod}(G(K)_{\mathrm{Ran}})$ is the δ-sheaf supported at $G(O)_{\mathrm{Ran}}$.

Remark 1.2.12. It also follows from definition that, any unital factorization algebra A internal to a factorization category automatically comes with a map of unital factorization algebras from the vacuum (as one expects).

Factorization Modules Now we let $\varphi : Z \to \mathrm{Ran}$ be a fixed map of prestacks, and consider the prestack

$$\mathrm{Ran}_{\varphi} : S \mapsto \{a : S \to Z, b : S \to \mathrm{Ran} \mid \varphi \circ a \subseteq b\}$$

this is called the φ-marked (or Z-marked) non-unital Ran space, which is a Ran-module object in the correspondence category of prestacks. Using it we can likewise form the notion of factorization modules / module categories / module spaces; the unital version also exists.

Example 1.2.13. Let's again be explicit. Suppose $\varphi : \{x\} \to \mathrm{Ran}$ is the inclusion of a single closed point along $\{x\} \to X \xrightarrow{\Delta} \mathrm{Ran}$. We often write Ran_x for Ran_{φ} in this case. Suppose $\mathcal{M}$ is an $\mathcal{A}$-factorization module. Then:

- The lowest strata is a single point $\{x\}$, the fiber at which is a vector space $M := \mathcal{M}|_x$;
- The second strata is $\{x\} \times X$, on it we have $\mathcal{M}|_{\{x\}\times X}$. The factorization condition says that on the disjoint locus $\mathcal{M}|_{\{x\}\times(X-x)}$, this sheaf is isomorphic to $M \otimes \mathcal{A}|_{X-x}$;
- Higher strata fibers look like $M \otimes \mathcal{A}|_{x_1} \otimes \ldots \otimes \mathcal{A}|_{x_n}$ for $x \neq x_1 \ldots \neq x_n$.

Remark 1.2.14. Here we see an inconvenience of the factorization structure: the module and the algebra are mixed together over all strata. This in particular means operations such as *restriction of factorization modules* is difficult to perform homotopy-coherently and requires care.

Proposition 1.2.15. *Let $\mathcal{M}$ be an unital factorization module category over some $\mathcal{C}$ at a point*[2] *x. Let $\mathbb{V}$ be the vacuum for $\mathcal{A}$. Then*

$$\mathcal{M}|_x \simeq \mathbb{V}\text{-}\mathrm{FactMod}_{\mathrm{un}}(\mathcal{M}).$$

[2]This is also true for more general marking φ.

Proof. Obviously right-to-left there is a restriction map. Conversely, suppose we have $M \in \mathcal{M}|_x$. The category $\mathcal{M}$ being an unital module category means that there is a unital map $\mathcal{M}|_x \to \mathcal{M}|_{\{x\}\times X}$; we stipulate that the corresponding factorization module's section over $\{x\} \times X$ is the image of M under this map. With more homotopy care one sees this provides an inverse. □

1.3 Conformal Blocks

Let $\mathcal{F}$ be a non-unital factorization algebra, and let's assume X is proper. Then the obvious pullback $\mathrm{Shv}(\mathrm{pt}) \to \mathrm{Shv}(\mathrm{Ran}(X))$ admits a left adjoint $\mathcal{F} \mapsto C_*^{\mathrm{ch}}(\mathcal{F})$, called the *conformal block*[3] a.k.a. factorization homology a.k.a. chiral homology of $\mathcal{F}$. Explicitly, the global sections $\Gamma(X^I, \mathcal{F}|_{X^I})$ form a $\mathsf{Comm_{nu}}^{\mathrm{op}}$-system, and we have

$$C_*^{\mathrm{ch}}(\mathcal{F}) \simeq \mathrm{colim}_{I \in \mathsf{Comm_{nu}}^{\mathrm{op}}} \Gamma(X^I, \mathcal{F}|_{X^I}).$$

Now we consider the *unital* case. The left adjoint to $\mathrm{Shv}(\mathrm{pt}) \to \mathrm{Shv}(\mathrm{Ran_{un}}(X))$ is again defined, so it makes sense to write conformal blocks for unital factorization algebras. However, unlike the non-unital case, we a priori have no good formula to compute this—that is, other than the tautological ones given by

$$\mathrm{colim}_{S \in \mathrm{Sch}_{/\mathrm{Ran_{un}}}} \Gamma_c(S, \mathcal{F}|_S).$$

Remark 1.3.1. Readers should bear in mind that lax prestacks are borderline non-geometric objects. To quote Dennis, "When dealing with lax prestacks, the conventional geometric intuition had better be abandoned for reasons of safety".

But in fact, we do have a way to compute, though this is not trivial:

Proposition 1.3.2. *Let $u : \mathrm{Ran} \to \mathrm{Ran_{un}}$ denote the obvious inclusion. Then we have*

$$C_*^{\mathrm{ch}}(\mathcal{F}) \simeq C_*^{\mathrm{ch}}(u^!(\mathcal{F})).$$

This follows from the fact that the map u is *universally homologically left cofinal* (uhlc), which is [Gai15b, Theorem 4.6.2]. The notion of uhlc is a bizarre one and does not appear in usual algebraic geometry. However, when $\mathcal{Y}_1 \to \mathcal{Y}_2$ is a map of *prestacks*, being uhlc coincides with being *universally homologically contractible* (uhc), which is defined as follows:

Definition 1.3.3. A map $\mathcal{Y}_1 \to \mathcal{Y}_2$ from a lax prestack to a prestack is uhc if, for any scheme S mapping to $\mathcal{Y}_2$, the corresponding pullback map

$$\mathrm{Shv}(S) \to \mathrm{Shv}(S \times_{\mathcal{Y}_2} \mathcal{Y}_1)$$

is fully faithful.

Intuitively, this means the fibers of the map are all contractible spaces, upon any base change.

Definition 1.3.4. A map $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ between lax prestacks is uhlc if, for any map $y : S \to \mathcal{Y}_2$ from a scheme, the "lax" base change prestack

$$\mathcal{Y}_1 \times_{\mathcal{Y}_2, \mathrm{lax}} S := ((z : T \to S) \mapsto \{y' \in \mathcal{Y}_1(T), y \circ z \Rightarrow f \circ y'\})$$

is uhc over S.

[3] Really should be its linear dual, but let's ignore that.

Being uhlc is not as strong as uhc, but it still guarantees the *global section* stays the same upon pulling back (i.e. the content of the previous proposition).

To convince our readers that the statement above is not so tautological, consider the following description of the left adjoint $u_!$:

Proposition 1.3.5. *If $\mathcal{F}$ is a non-unital factorization algebra, then for a point $s \in \mathrm{Ran}_{\mathrm{un}}(X)(S)$ such that the corresponding I-tuples of maps $S \to X$ land in the disjoint locus, the corresponding section $u_!(\mathcal{F})|_s$ is given by*

$$\bigoplus_{J \subseteq I} \mathcal{F}|_{\alpha_J}$$

where $\alpha_J \in \mathrm{Ran}(X)(S)$ is the subset corresponding to $J \subseteq I$.

Propagation Methods Consider the laft lax prestack

$$\mathrm{Arrow}(\mathrm{Ran}_{x,\mathrm{un}}) := S \mapsto \mathrm{Fun}([2], \mathrm{Ran}_{x,\mathrm{un}}(S)).$$

We have two projections

$$p_s, p_t : \mathrm{Arrow}(\mathrm{Ran}_{x,\mathrm{un}}) \to \mathrm{Ran}_{x,\mathrm{un}}$$

"remembering" respectively the source and the target. Note that we have a 2-morphism $\mathfrak{i} : p_s \to p_t$. We equip $\mathrm{Arrow}(\mathrm{Ran}_{x,\mathrm{un}})$ with the factorization module space structure (for the factorization space $\mathrm{Ran}_{\mathrm{un}}$) such that p_t is compatible with the factorization module structures but p_s is not. For a map $y : \mathrm{pt} \to \mathrm{Ran}_{\mathrm{un}}$ corresponding to a finite subset J with $I_t \cap J = \varnothing$, the map

$$(\mathrm{Ran}_{\mathrm{un}} \times \mathrm{Arrow}(\mathrm{Ran}_{x,\mathrm{un}}))_{\mathrm{disj}} \to \mathrm{Arrow}(\mathrm{Ran}_{x,\mathrm{un}})$$

sends (y, z) to the point corresponding to $I_0 \subset I_s \subset J \sqcup I_t$.

Remark 1.3.6. Explicitly, a map $z : \mathrm{pt} \to \mathrm{Arrow}(\mathrm{Ran}_{x,\mathrm{un}})$ corresponds to a chain $I_0 \subset I_s \subset I_t$ of finite subsets of $\mathrm{Maps}(\mathrm{pt}, X)$ such that $I_0 \subset I_s$ (resp. $I_0 \subset I_t$) corresponds to the composition $\mathrm{pt} \to \mathrm{Arrow}(\mathrm{Ran}_{x,\mathrm{un}}) \xrightarrow{p_s} \mathrm{Ran}_{x,\mathrm{un}}$ (resp. $\mathrm{pt} \to \mathrm{Arrow}(\mathrm{Ran}_{x,\mathrm{un}}) \xrightarrow{p_t} \mathrm{Ran}_{x,\mathrm{un}}$).

For any unital factorization category $\mathcal{A}$ over $\mathrm{Ran}_{\mathrm{un},\mathrm{dR}}$, and for any factorization $\mathcal{A}$-module $\mathcal{C}$ over $\mathrm{Ran}_{x,\mathrm{un},\mathrm{dR}}$, the (co)restriction $\mathbf{cores}_{p_{t,\mathrm{dR}}}(\mathcal{C})$, which is a crystal of categories over $\mathrm{Arrow}(\mathrm{Ran}_{x,\mathrm{un}})_{\mathrm{dR}}$, inherits a factorization $\mathcal{A}$-module structure. Moreover, for any unital factorization algebra A in $\mathcal{A}$, there is a canonical functor

$$\mathrm{prop}_A : A\text{-}\mathrm{FactMod}_{\mathrm{un}}(\mathcal{C}) \to A\text{-}\mathrm{FactMod}_{\mathrm{un}}(\mathbf{cores}_{p_{t,\mathrm{dR}}}(\mathcal{C}))$$

such that the following diagram commutes

$$\begin{tikzcd} A\text{-}\mathrm{FactMod}_{\mathrm{un}}(\mathcal{C}) \arrow[r, "\mathrm{oblv}"] \arrow[d, "\mathrm{prop}_A"] & \mathbf{\Gamma}(\mathrm{Id}_{\mathrm{dR}}, \mathcal{C}) \arrow[d, "\mathrm{res}"] \\ A\text{-}\mathrm{FactMod}_{\mathrm{un}}(\mathbf{cores}_{p_{t,\mathrm{dR}}}(\mathcal{C})) \arrow[r, "\mathrm{oblv}"] & \mathbf{\Gamma}(p_{t,\mathrm{dR}}, \mathcal{C}) \end{tikzcd}$$

where the bottom functor is the restriction functor for subsections of $\mathcal{C}$. More generally, for any morphism $f : A \to B$ between unital factorization algebras, we can use restriction of factorization modules (however they are defined) to define the composition

$$\mathrm{prop}_A : B\text{-}\mathrm{FactMod}_{\mathrm{un}}(\mathcal{C}) \xrightarrow{\mathrm{prop}_B} B\text{-}\mathrm{FactMod}_{\mathrm{un}}(\mathbf{cores}_{p_{t,\mathrm{dR}}}(\mathcal{C})) \xrightarrow{\mathrm{res}_f} A\text{-}\mathrm{FactMod}_{\mathrm{un}}(\mathbf{cores}_{p_{t,\mathrm{dR}}}(\mathcal{C}))$$

and have a natural transformation

$$\begin{tikzcd}
B\text{-FactMod}_{\mathrm{un}}(\mathcal{C}) \arrow[r, "\mathrm{oblv}"] \arrow[d, "\mathrm{prop}_A"] \arrow[dr, phantom, "\Nearrow"] & \mathbf{\Gamma}(\mathrm{Id}_{\mathrm{dR}}, \mathcal{C}) \arrow[d, "\mathrm{res}"] \\
A\text{-FactMod}_{\mathrm{un}}(\mathbf{cores}_{p_{t,\mathrm{dR}}}(\mathcal{C})) \arrow[r, "\mathrm{oblv}"] & \mathbf{\Gamma}(p_{t,\mathrm{dR}}, \mathcal{C})
\end{tikzcd} \tag{1}$$

Construction 1.3.7. In the special case when A = unit is the unit factorization algebra of $\mathcal{A}$, we have the following commutative diagram

$$\begin{tikzcd}
B\text{-FactMod}_{\mathrm{un}}(\mathcal{C}) \arrow[r, "\mathrm{oblv}"] \arrow[d, "\mathrm{prop}_{\mathrm{unit}}"] & \mathbf{\Gamma}(\mathrm{Id}_{\mathrm{dR}}, \mathcal{C}) \arrow[d, "\mathrm{ins}_{\mathrm{unit}}"] \\
\mathrm{unit}\text{-FactMod}_{\mathrm{un}}(\mathbf{cores}_{p_{t,\mathrm{dR}}}(\mathcal{C})) \arrow[r, "\mathrm{oblv}"] & \mathbf{\Gamma}(p_{t,\mathrm{dR}}, \mathcal{C})
\end{tikzcd}$$

where $\mathrm{ins}_{\mathrm{unit}}$ is the functor of *unit insertion*

$$\mathrm{ins}_{\mathrm{unit}} : \mathbf{\Gamma}(\mathrm{Id}_{\mathrm{dR}}, \mathcal{C}) \xrightarrow{\mathrm{res}} \mathbf{\Gamma}(p_{s,\mathrm{dR}}, \mathcal{C}) \xrightarrow{\mathrm{i}_!} \mathbf{\Gamma}(p_{t,\mathrm{dR}}, \mathcal{C}).$$

Digression: what is propagation? Let us explain the name. Suppose A is a unital factorization algebra and N is its, such that its fiber at a $\mathbb{C}$-point

$$y := [\{x\} \subseteq \{x, x_1, \ldots, x_n\}] \in \mathrm{Arrow}(\mathrm{Ran}_{x,\mathrm{un}})(\mathbb{C})$$

is given by some vector space $N_0 \otimes A_0^{\otimes(n-1)}$. The map y induces a map

$$(\mathrm{Ran}_{\mathrm{un},y} := S \mapsto \{\alpha : S \to \mathrm{Ran}_{\mathrm{un}} \mid (x, x_1, \ldots, x_n) \subseteq \alpha\}) \to \mathrm{Arrow}(\mathrm{Ran}_{x,\mathrm{un}})$$

Then the pullback of $\mathrm{prop}_A(N)$ along this map produces a unital A-factorization module on the former, which should be thought of as consisting of $(N_0, A_0, \ldots, A_0)$ supported at $(x, x_1, \ldots, x_n)$ respectively. Thus, it can be seen as "propagating" the algebra A to other points on the curve.

Another perspective is as follows. As explained in [Ras15], via étale hyperdescent the category of A-unital factorization modules in C *itself* form a weak unital factorization category under *chiral fusion*, and the underlying sheaf of $\mathrm{prop}_A(N)$ can be seen as observing A as an element $A^{\mathrm{enh}} \in \mathrm{FactAlg}_{\mathrm{un}}(A\text{-FactMod}_{\mathrm{un}}(\mathcal{C}))$ and N as an element $N^{\mathrm{enh}} \in A^{\mathrm{enh}}\text{-FactMod}_{\mathrm{un}}(A\text{-FactMod}_{\mathrm{un}}(\mathcal{C}))$. This is, of course, evocative of the "enhancement" equivalences

$$\mathrm{AlgEnh} : \mathcal{O}\text{-Alg}(\mathcal{C})_{A/} \simeq \mathcal{O}\text{-Alg}(A\text{-mod}^{\mathcal{O}}(\mathcal{C}))$$

$$\mathrm{ModEnh} : A\text{-mod}^{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{AlgEnh}(A)\text{-mod}^{\mathcal{O}}(A\text{-mod}^{\mathcal{O}}(\mathcal{C}))$$

which holds for any coherent ∞-operad $\mathcal{O}$, any (weakly) $\mathcal{O}$-monoidal category $\mathcal{C}$ and any $\mathcal{O}$-algebra $A \in \mathcal{C}$, as established in [Lur12, 3.4.1.7] and [Lur12, 3.4.1.9].

Let's consider $\mathcal{C} \simeq \mathrm{DMod}(\mathrm{Ran}_{\mathrm{un},x})$. Then the functor $\mathrm{res} : \mathbf{\Gamma}(\mathrm{Id}_{\mathrm{dR}}, \mathcal{C}) \to \mathbf{\Gamma}(\mathrm{p}_{t,\mathrm{dR}}, \mathcal{C})$ is just $p_t^! : \mathrm{DMod}(\mathrm{Ran}_{x,\mathrm{un}}) \to \mathrm{DMod}(\mathrm{Arrow}(\mathrm{Ran}_{x,\mathrm{un}}))$. Hence we obtain a natural transformation

$$\begin{tikzcd}
B\text{-FactMod}_{\mathrm{un}}(\mathrm{DMod}_{\mathrm{Ran}_{x,\mathrm{un}}}) \arrow[r, "\mathrm{oblv}"] \arrow[d, "\mathrm{prop}_A"] \arrow[dr, phantom, "\Nearrow"] & \mathrm{DMod}(\mathrm{Ran}_{x,\mathrm{un}}) \\
A\text{-FactMod}_{\mathrm{un}}(\mathrm{Arrow}(\mathrm{Ran}_{x,\mathrm{un}})) \arrow[r, "\mathrm{oblv}"] & \mathrm{DMod}(\mathrm{Arrow}(\mathrm{Ran}_{x,\mathrm{un}})) \arrow[u, "p_{t,!}"']
\end{tikzcd} \tag{2}$$

Diagram (2) does not commute in general. But we have the following results:

Lemma 1.3.8. *Diagram* (2) *commutes after taking chiral homology, i.e., after composing with*

$$\mathrm{C}^{\mathrm{ch}}_{x,\mathrm{un}} : \mathrm{DMod}(\mathrm{Ran}_{x,\mathrm{un}}) \to \mathrm{Vect}.$$

Recall $\mathrm{C}^{\mathrm{ch}}_{x,\mathrm{un}}$ is given by !-pushforward along $p : \mathrm{Ran}_{x,\mathrm{un}} \to \mathrm{pt}$. In the case when $B = A$, the following stronger result is true:

Lemma 1.3.9. *The composition*

$$p_! \circ p_{s,!} \circ p_t^! \simeq p_! \circ p_{t,!} \circ p_t^! \to p_!$$

in $\mathrm{Funct}(\mathrm{DMod}(\mathrm{Ran}_{x,\mathrm{un}})) \to \mathrm{Vect}$ *is an equivalence. Moreover, the natural transformation*

$$p_{s,!} \circ p_t^! \to p^! \circ p_!$$

is an equivalence.

Let us explain the names. For a set of unital A-factorization modules $N_1, \dots, N_k$ inserted at distinct points $x_1, \dots, x_k \in X(\mathbb{C})$, let us write the chiral homology as

$$\langle N_1, \dots, N_k \rangle^A_{x_1,\dots,x_k};$$

Given a unital factorization A-module M supported at x, the equality between fibers of $p_{s,!} \circ p_t^!(M)$ and $p^! \circ p_!(M)$ at some $(x, x_1, \dots, x_n)$, given by the propagation lemma for chiral homology, translates to

$$\langle M, A, \dots, A \rangle^A_{x,x_1,\dots,x_n} \simeq \langle M \rangle^A_x,$$

which says that when computing A-chiral homology, extra copies of A would not affect the result. The propagation-restriction lemma, on the other hand, roughly states that if M is a B-factorization module and $A \to B$ is a map between unital factorization algebras, then we have

$$\big\langle \langle M \rangle^A_x, \langle M, B \rangle^A_{x,x_1}, \langle M, B, B \rangle^A_{x,x_1,x_2}, \dots \big\rangle \simeq \langle M \rangle^B_x$$

where the LHS now ranges over all of $\mathrm{Ran}_{x,\mathrm{un}}$.

Remark 1.3.10. The punchline is that this method allows us to reduce questions regarding A-factorization modules for arbitrary unital factorization algebra A to that of vacuum-factorization modules, which, per Proposition 1.2.15, is easier to handle.

We refer readers to the main paper for the proofs of these statements. Both boil down to some uhlc statements, and as such, are *only* valid in the unital setting.

2 Talk 2: Koszul, Verdier, and all that

The word "Koszul duality" can mean many things, but for us it means an operaidc generalization of the Bar-coBar duality for associative algebras. A quick summary of the general formulation of operadic Koszul duality can be found in [Fu]; I'll cover some of it in my talk, and here will assume that people have already read *op.cit.* (Not everything in there is needed though.)

Remark 2.0.1. The main thing to keep in mind when using Koszul dualities is that of *convergence* issue: only things with strong nilpotency conditions can be perfectly preserved by Koszul duality. When there is an extra grading (e.g. in the graded factorization algebra case, or the case of mixed geometry), this problem can be alleviated using the so-called "red-shift trick" (c.f. [AG15, Appendix A.2]).

Digression: Chiral Koszul Duality In the case of Vect-valued non-unital factorization algebras (whose underlying sheaf is simply a D-module on $\mathrm{Ran}(X)$), we can consider the category $\mathrm{DMod}(\mathrm{Ran}(X))_{\mathrm{ch}}$, which is $\mathrm{DMod}(\mathrm{Ran}(X))$ equipped with the *chiral* symmetric monoidal tensor, which is pull-push along the correspondence diagram $\mathrm{Ran}(X)\times\mathrm{Ran}(X) \leftarrow [\mathrm{Ran}(X)\times\mathrm{Ran}(X)]_{\mathrm{disj}} \to \mathrm{Ran}(X)$. Then it makes sense to discuss the Koszul adjunction

$$\mathrm{LieAlg}(\mathrm{DMod}(\mathrm{Ran}(X))_{\mathrm{ch}}) \rightleftarrows \mathrm{CoCommCoAlg}(\mathrm{DMod}(\mathrm{Ran}(X))_{\mathrm{ch}})$$

It was proven in [FG12] that this identifies those Lie-algebra objects supported on the main diagonal with those cocommutative coalgebras having a factorization property. The former is what we usually call *chiral algebras*, but note this definition also works for higher dimension.

2.1 $\mathbb{E}_2$-algebras

Recall the topological operad of (unframed) little 2-disks $\mathbb{E}_2$. For any symmetric monoidal category $\mathcal{C}$, it then makes sense to discuss $\mathbb{E}_2$-objects within it. For instance, if we take DGCat, we obtain the notion of $\mathbb{E}_2$ DG categories, which is a derived generalization of *braided monoidal* abelian categories. If $\mathcal{C}$ is stable, then the action of $\mathbb{E}_2$ operad factors through its stabilization; if $\mathcal{C}$ is Vect-enriched (i.e. a DG category), then it factors through the operad $\{C_*(\mathbb{E}_2(n))\}_{n\geq 1}$.

Remark 2.1.1. In fact, to discuss $\mathbb{E}_n$-objects in category $\mathcal{C}$ it suffices to have $\mathcal{C}$ itself be $\mathbb{E}_n$; in other words, *twisted* $\mathbb{E}_2$-algebra objects internal to an $\mathbb{E}_2$-category makes sense. Note that however $\mathbb{E}_2$-objects in an $\mathbb{E}_1$-category doesn't make sense.

We note that:

- Dunn additivity says $\mathbb{E}_2 \simeq \mathbb{E}_1 \otimes \mathbb{E}_1$, which concretely means that for any $\mathbb{E}_2$-category $\mathcal{C}$ we have

  $$\mathbb{E}_2\text{-Alg}(\mathcal{C}) \simeq \mathbb{E}_1\text{-Alg}(\mathbb{E}_1\text{-Alg}(\mathcal{C}))$$

  the similar claim holds for the non-unital version as well;

- Recall from the notes above that $\mathbb{E}_n$-operads are self-dual up to a cohomological shift. Alternatively[4], we can use the adjunction

  $$\mathrm{Bar}^2 : \mathbb{E}_2\text{-Alg}_{\mathrm{aug}}(\mathcal{C}) \rightleftarrows \mathbb{E}_2\text{-CoAlg}_{\mathrm{coaug}}(\mathcal{C}) : \mathrm{coBar}^2$$

  i.e. applying bar construction twice (here the subscript means "(co)augmented".) This adjunction factors as

  $$\mathbb{E}_2\text{-Alg}_{\mathrm{aug}}(\mathcal{C}) \rightleftarrows \mathrm{BiAlg}_{\mathrm{biaug}}(\mathcal{C}) := \mathbb{E}_1\text{-Coalg}_{\mathrm{coaug}}(\mathbb{E}_1\text{-Alg}_{\mathrm{aug}}(\mathcal{C})) \rightleftarrows \mathbb{E}_2\text{-CoAlg}_{\mathrm{coaug}}(\mathcal{C})$$

  We will refer to $\mathrm{Bar}^2 \rightleftarrows \mathrm{coBar}^2$ as $\mathbb{E}_2$ Koszul duality. Using the fact that taking direct sum with $\mathbb{C}$ is an equivalence between non-unital $\mathbb{E}_n$-algebras and augmented (unital) $\mathbb{E}_n$-algebras, we extend this adjunction to non-unital algebras.

Remark 2.1.2. Thus we see that $\mathbb{E}_2$-algebras are, up to convergence issues, roughly the same as bialgebras. Similarly, $\mathbb{E}_2$-modules are roughly the same as Yetter-Drinfeld modules.

[4] i.e. after shifting by 2 it agrees with the operadic version. The advantage of this iterated bar version is that it makes sense even for the unstable setting.

2.2 Factorization Cosheaf

At this point it is useful to compare the notion above with the notion of factorization algebras used by topologists. We will fix a symmetric monoidal category $\mathcal{C}$.

Definition 2.2.1. Set $\mathcal{R} := \mathrm{Ran}(X)(\mathbb{C})$. For each finite collection of disjoint open subsets $\{U_i\}$ of $X(\mathbb{C})$ (equipped with the analytical topology), let $\mathrm{Ran}(\{U_i\})$ denote the collection of subsets S of $X(\mathbb{C})$ such that S contains at least one element from each U_i and is contained in $\bigcup_i U_i$. Then $\mathcal{R}$ can be equipped with the *coarest* topology such that $\mathrm{Ran}(\{U_i\})$ is open for each $\{U_i\}$; this we call the *topological* Ran space.

Convention 2.2.2. By a $\mathcal{C}$-valued cosheaf on $\mathcal{R}$ we will always mean a *hypercocomplete*[5] cosheaf on this topological space.

Definition 2.2.3. There is an operad $\mathrm{Fact}(\mathcal{R})$ given as follows:

- The objects are finite sequences $(U_1, \dots, U_j)$ of open subsets of $\mathcal{R}$;
- a map $(U_1, \dots, U_j) \to (V_1, \dots, V_k)$ is a surjection $\pi : j \twoheadrightarrow k$ such that for each r, $\{U_i\}_{\pi(i)=r}$ is pairwise independent, and $\star_{\pi(i)=r} U_i \subseteq V_r$. Here $A \star B := \{S \cup T \mid S \in A, T \in B\}$.

Definition 2.2.4. A $\mathcal{C}$-valued topological factorization cosheaf is a $\mathrm{Fact}(\mathcal{R})$-algebra $\mathcal{F}$ in $\mathcal{C}$ such that the restriction to arity 1 produces a cosheaf in the sense above, and such that, for every $U, V \subseteq \mathcal{R}$ such that $\bigcup_{U_0 \in U} U_0 \cap \bigcup_{V_0 \in V} V_0 = \varnothing$, the natural map $\mathcal{F}(U) \otimes \mathcal{F}(V) \to \mathcal{F}(U \star V)$ (coming from the operad structure) is an isomorphism.

Here's why this notion is interesting:

- On one hand, for $X = \mathbb{A}^1$ and *any* $\mathcal{C}$, a theorem of Lurie tells us that the category of non-unital $\mathbb{E}_2$-algebras in $\mathcal{C}$ is equivalent to the subcategory of factorization cosheaves consisting of those objects which are locally constant on each stratum (more explicitly, for such an $\mathbb{E}_2$-algebra A and any disjoint set of disks $\{U_i\}$ of cardinality s, we declare $\mathcal{F}_A(\mathrm{Ran}(\{U_i\}) := A^{\otimes s}$, and then left Kan extend to all open subsets);
- On the other hand, for $\mathcal{C}$ *stable*, Verdier duality tells us that the category of factorization cosheaves is equivalent to the category of factorization algebras as in the previous section, if we take "Shv" to mean "topological !-sheaves".

Now, if one ignores homotopy coherence (which one cannot), it is not hard to directly demonstrate the procedure of producing a !-sheaf from an $\mathbb{E}_2$-algebra A. Let's do the section over $\mathbb{A}^2$. The resulting sheaf will be a sheaf $\mathcal{F}$ on the diagonal $\mathbb{A}^1$ boxed by the dualizing sheaf along the anti-diagonal direction, so we'll describe $\mathcal{F}$ instead. Over $\mathbb{G}_m$ it is $A^2 \otimes \omega_{\mathbb{G}_m} =: j^!(\mathcal{F})$, and over the central point it is $A =: i^!(\mathcal{F})$. Then, to construct $\mathcal{F}$ it suffices to supply a nearby cycle map

$$\varphi : i^! j_! j^!(\mathcal{F}) \to i^!(\mathcal{F});$$

[5] This is a technical point to bear in mind, but a discussion would lead us too far afield. Essentially the problem is as follows: there is a natural stratification $\{\mathcal{R}_{\le k}\}_{k \in \mathbb{N}}$ on $\mathcal{R}$ given by cardinality, but the map $\mathrm{colim}_k \mathcal{R}_{\le k} \to \mathcal{R}$ is not an isomorphism of topological spaces. What we should consider is the category of cosheaves on the former space, i.e. those cosheaves such that $\mathcal{F} \to \lim_k (i_k)_\dagger (i_k)^\dagger \mathcal{F}$ is an isomorphism, where $i_k : \mathcal{R}_{\le k} \to \mathcal{R}$ is the natural map.

indeed, given such a map, we can recover $\mathcal{F}$ as

$$\mathrm{Cone}(i_! \mathrm{Cocone}(\varphi) \to i_! i^! j_! j^! (\mathcal{F}) \simeq j_! j^! (\mathcal{F}))$$

So what is our choice of φ? By design our sheaf is $\mathbb{G}_m$-equivariant, so by contraction principle the above map φ is of the shape $C_*(\mathbb{G}_m) \otimes A^2 \to A$, and we use the multiplication map provided by the $\mathbb{E}_2$-structure.

Remark 2.2.5. The construction has the feature such that, for a point $\overline{x} = (x_1, \ldots, x_k)$ of k-disjoint points, the !-fiber at it of the resulting factorization !-sheaf is given by (the chain complex underlying) $A^{\otimes k}$, where A is the non-unital $\mathbb{E}_2$-algebra we started with.

Koszul as Verdier On compact objects, $\mathrm{Shv}(\mathrm{Ran}(X))$ has a (contravariant) Verdier self-duality $\mathbb{D}$. It is uniquely defined such that it is compatible with Verdier duality on each X^I under the !-*pushforward* functors. This isn't a super well-behaved functor: for instance we have $\mathbb{D}(\omega_{\mathrm{Ran}(X)}) \simeq 0$.

However, the key fact for us is that, if we let $\mathcal{F}_A$ denote the sheaf corresponding to A (for simplicity, let us assume it is a compact object in Vect), then

$$\mathcal{F}_{\mathrm{Bar}^2(A)^\vee} \simeq \mathbb{D}(\mathcal{F}_A)$$

where $\vee$ is the linear dual. In other words, $\mathbb{E}_2$ Koszul duality translates to Verdier duality on the Ran space.

Note that these two statements are compatible: indeed, the $\mathbb{E}_2$-algebra corresponding to $\omega_{\mathrm{Ran}(X)}$ is the algebra $\mathbb{C}$ (considered as a non-unital $\mathbb{E}_2$-algebra by forgetting that it was unital), and its Koszul dual is zero.

Remark 2.2.6. Since Verdier duality switches $*$-fibers and !-fibers, intuitively the "$*$-fiber" of $\mathcal{F}_A$ should be given by powers of the $\mathbb{E}_2$ Koszul dual of A. Now this doesn't literally make sense, because $*$-fibers are not well-defined for sheaves on the Ran space. However, if we had an extra grading, then the *graded* Ran space (a.k.a. the configuration space) splits into many schematic pieces, on which this intuition holds true literally.

What About Bialgebras? Here is a valid question to ask: what is the functor corresponding to the operation

$$\mathrm{FactAlg}_{\mathrm{loc.const.}}(\mathrm{Shv}(\mathrm{Ran}(\mathbb{R}^2))) \simeq \mathbb{E}_2\text{-}\mathrm{Alg}_{\mathrm{nu}}(\mathrm{Vect}) \xrightarrow{\mathrm{Bar}} \mathrm{BiAlg}(\mathrm{Vect}) \xrightarrow{\mathrm{oblv}} \mathrm{Vect}$$

Since we saw above that taking 2 bars correspond to switch from !-fiber to $*$-fiber, it should not come as no surprise that we should expect a mixture of these two. Indeed, the operation

$$\mathbb{E}_2\text{-}\mathrm{Alg}_{\mathrm{nu}}(\mathrm{Vect}) \xrightarrow{\mathrm{Bar}} \mathrm{BiAlg}(\mathrm{Vect}) \xrightarrow{\mathrm{oblv}} \mathrm{Vect}$$

can be rewritten as

$$\mathbb{E}_2\text{-}\mathrm{Alg}_{\mathrm{nu}}(\mathrm{Vect}) \xrightarrow{\mathrm{oblv}} \mathbb{E}_1\text{-}\mathrm{Alg}_{\mathrm{nu}}(\mathrm{Vect}) \xrightarrow{\mathrm{oblv} \circ \mathrm{Bar}} \mathrm{Vect}$$

and the forgetful functor from $\mathbb{E}_2$ to $\mathbb{E}_1$ corresponds to restricting along the inclusion $\mathrm{Ran}(\mathbb{R}) \subset \mathrm{Ran}(\mathbb{R}^2)$ induced by the inclusion of a line $\mathbb{R} \subseteq \mathbb{R}^2$ (it doesn't matter which line we choose). In other words, we should expect the operation to be

$$\mathrm{FactAlg}_{\mathrm{loc.const.}}(\mathrm{Shv}(\mathrm{Ran}(\mathbb{R}^2))) \xrightarrow{!\text{-pull}} \mathrm{FactAlg}_{\mathrm{loc.const.}}(\mathrm{Shv}(\mathrm{Ran}(\mathbb{R}))) \xrightarrow{*\text{-pull}} \mathrm{Vect}$$

(Again, to make actual sense of this one needs an extra grading.) This is what is usually known as taking *hyperbolic stalks*.

Now, a priori, hyperbolic stalk with respect to the *real* strata is not an operation one can do in AG. However, as was shown in Kashiwara-Shapira (and certainly well known before then), for the coordinate stratification such hyperbolic stalks can be computed via *vanishing cycles*. More precisely, say we have a constructible sheaf $\mathcal{F}$ on $\mathbb{C}^n$ with respect to the coordinate stratification, and let $f : \mathbb{C}^n \to \mathbb{C}$ be $\sum_i z_i^2$. Consider the inclusions $\{0\} \xrightarrow{i} \mathbb{R}^n \xrightarrow{j} \mathbb{C}^n$, then we have $i^* j^! \mathcal{F} \simeq \Phi_f(\mathcal{F})|_0$, where RHS is the stalk of $\Phi_f(\mathcal{F})$ at $\{0\} \subseteq \{f = 0\}$ (in fact the vanishing cycle is concentrated on this point). (Details can be found in [KS19].) This gives rise to the description in [BFS06] of the above functor in terms of vanishing cycles.

Now, despite this description being slightly awkward (from a derived perspective), there are some advantages to it:

- Many factorization algebras in real life become *abelian* bialgebras, so that the corresponding factorization algebras are *perverse sheaves*. This significantly reduces the amount of data required (higher compatibilities become conditions);
- Note that the operation of $\mathbb{E}_2$ Koszul duality simply becomes linear dual if we flip both sides by 1 bar operation; this corresponds to the well-known fact that vanishing cycles commute with taking Verdier dual.

2.3 Categorical Verdier

We now address the key question left from above: if $\mathcal{C}$ is *not* stable, how does one convert from factorization cosheaves to factorization sheaves? More specifically, consider $\mathcal{C} \simeq \mathrm{DGCat}$. Let us work in a more general paradigm. Fix some reasonable topological space X (up to the problem of hypercompletion, it's okay to take X to be the topological Ran space), which comes with a stratification $X \to A$ where A is some poset. For every $\mathcal{C}$ let $\mathrm{cShv}_A(X; \mathcal{C})$ denote the category of $\mathcal{C}$-valued cosheaves. Our goal is to construct a functor

$$\Theta : \mathrm{cShv}_A(X; \mathrm{DGCat}) \to \mathrm{cShv}_A(X; \mathrm{Vect})\text{-mod}(\mathrm{DGCat})$$

This functor should have the following properties:

- It is compatible with base-change to lower open/closed strata;
- When A is trivial, the corresponding map (here LS denotes the category of local systems)

$$\mathsf{LS}(X; \mathrm{DGCat}) \to \mathsf{LS}(X; \mathrm{Vect})\text{-mod}(\mathrm{DGCat})$$

  has the underlying category being the global section (defined as the left adjoint of the pullback functor) of the local system.

Remark 2.3.1. Let us recall that if $\mathcal{F}$ is a local system of DG categories, then its global section $F := \Gamma(X, \mathcal{F})$ automatically has an action of $\mathsf{LS}(X; \mathrm{Vect})$ which is "fiberwise". Let's make this explicit. Let $\pi : X \to \mathrm{pt}$ be the projection, then it is clear that $\pi_\dagger \pi^\dagger(C) \simeq \mathsf{LS}(X) \otimes C$. It follows then that the global section always has a $\mathsf{LS}(X)$-comodule structure. However, note that $\mathsf{LS}(X)$ is self-dual as a DG category, so we also obtain a $\mathsf{LS}(X)$-module structure.

Even more explicitly, the action is given by

$$\mathsf{LS}(X)\otimes\Gamma(X,\mathcal{F})\simeq(\mathrm{colim}_{X^{\mathrm{op}}}\mathrm{Vect})\otimes\Gamma(X,\mathcal{F})\simeq\mathrm{colim}_{(x\in X^{\mathrm{op}},y\in X^{\mathrm{op}})}\mathcal{F}_y\simeq\lim_{(x,y)\in X\times X}\mathcal{F}_y\to\lim_{y\in X}\mathcal{F}_y$$

given by restriction to the diagonal.

If the functor Θ is given, then our strategy is clear: for each $X^I(\mathbb{C})$, we declare that the corresponding section is

$$\mathrm{Shv}(X^I)\otimes_{\mathrm{cShv}_A(X^I)}\Theta(\mathcal{F}|_{X^I})$$

Here the action of $\mathrm{cShv}_A(X^I)$ on $\mathrm{Shv}(X^I)$ is via Verdier duality; it will then formally follow that this has a factorization category structure. So let's focus on constructing Θ.

Suppose we *were* working in the stable setting, working with the usual open-closed stratification $U\xrightarrow{j}X\xleftarrow{i}Z$, then the Cousin triangle tells us that the category $\mathrm{Shv}(X)$ is equivalent to the category of triples

$$\{\mathcal{F}_U\in\mathrm{Shv}(U),\mathcal{F}_Z\in\mathrm{Shv}(Z),\varphi:i^*j_*(\mathcal{F}_U)\to\mathcal{F}_Z\}$$

i.e. to the *lax* limit of the diagram $\mathrm{Shv}(U)\xrightarrow{i^*j_*}\mathrm{Shv}(Z)$. This description works if we replace $\bullet\to\bullet$ with any *reasonable* stratification $X\to A$. The basic observation is that, for constructible sheaves, this works in the unstable setting as well:

Proposition 2.3.2. *For reasonable stratification $X\to A$ and any symmetric monoidal $\mathcal{C}$, the category $\mathrm{cShv}_A(X;\mathcal{C})$ is isomorphic to the lax limit of the diagram $\mathsf{LS}(X_a;\mathcal{C})$ connected by the †-push-†-pull (a.k.a. nearby cycle) morphisms.*

The following can be checked by unfolding definitions:

- A diagram as in the above proposition induces a diagram $\mathscr{D}$ of the global sections of the components;
- Moreover, the lax limit of $\mathscr{D}$ automatically carries an action of $\mathrm{cShv}_A(X;\mathrm{Vect})$.

Thus we can define our Θ to be this lax limit. Of course, this is not an appropriate way to define functors; we refer readers to our paper for the actual definition.

It follows from construction that (internal) $\mathbb{E}_2$-algebras and $\mathbb{E}_2$-modules embed fully faithfully as constructible factorization (module) sheaves under this categorical Verdier duality.

3 Talk 3: Renormalized Ind-coherent Sheaves

The main technical bedrock of our geometric approach to affine Lie algebra representations is the theory of *factorizable renormalized ind-coherent sheaves*, established in the appendix of [CF21]. The main output of said theory can be summarized in the following claim:

Claim 3.0.1. *For a class of sufficiently nice prestacks $\mathcal{Y}$, there exists a compactly generated DG category $\mathrm{IndCoh}_{*,\mathrm{ren}}(\mathcal{Y})$ equipped with a (not necessarily left complete) t-structure, such that the bounded below category coincides with that of $\mathrm{QCoh}(\mathcal{Y})$. We let $\mathrm{IndCoh}^{!,\mathrm{ren}}(\mathcal{Y})$ denote its dual category.*

Let B_{dR} be the de Rham prestack associated with a laft prestack B (we shall mostly consider $B = \mathrm{Ran}$). Suppose $\mathcal{Y} \to B_{\mathrm{dR}}$ is a prestack whose fiber at every $x : \mathrm{pt} \to B_{\mathrm{dR}}$ is reasonably nice. Then there exists a crystal of categories $\mathcal{I}nd\mathcal{C}oh_{*,\mathrm{ren}}(\mathcal{Y})$ *(and a dual version $\mathcal{I}nd\mathcal{C}oh^{!,\mathrm{ren}}(\mathcal{Y})$) over B_{dR}, whose fiber at every x is given by the renormalized ind-coherent sheaf category* $\mathrm{IndCoh}_{*,\mathrm{ren}}(Y_x)$ *(resp.* $\mathrm{IndCoh}^{!,\mathrm{ren}}(Y_x)$*).*

Convention 3.0.2. In this section, all occurrences of PreStk and the term "prestack" should be understood as *convergent prestack*. For a prestack $\mathcal{Y}$ to be convergent means that, for every affine S, we require the natural map $\mathcal{Y}(S) \to \lim_n \mathcal{Y}(\tau^{\leq n}(S))$ (where $\tau^{\leq n}$ denotes the Postnikov truncation of S) to be an isomorphism. We note that being convergent is a condition that is satisfied by almost all "geometric" objects (e.g. all Artin stacks).

3.1 Need For Renormalization

This subsection could be seen as an "user's guide" to [Ras20] and the appendix of [CF21].

The first question to consider is, what should be the category of representations for the arc group $\mathcal{L}^+G$? Unlike the finite-type case, $\mathrm{QCoh}(\mathbb{B}\mathcal{L}^+G)$ is not the right choice. Already in the classical representation theory of p-adic groups people knew that only *smooth* representations (those coming from a finite-dimensional quotient of $\mathcal{L}^+G$) should be considered; in more categorical language, the naïve choice $\mathrm{QCoh}(\mathbb{B}\mathcal{L}^+G)$ is "too pathologic" in the sense that it is not compactly generated.

The solution is to decree that we only care about whatever we can access using *finite-dimensional* representations. More precisely, let $\mathrm{QCoh}(\mathbb{B}\mathcal{L}^+G)^{\mathrm{fd}}$ be the subcategory spanned by finite-dimensional representations, and we declare

$$\mathrm{Rep}(\mathcal{L}^+G) \coloneqq \mathrm{Ind}(\mathrm{QCoh}(\mathbb{B}\mathcal{L}^+G)^{\mathrm{fd}});$$

i.e. the category of formal filtered colimits of finite-dimensional representations.

Remark 3.1.1. Readers who have met before the theory of ind-coherent sheaves ([GR17]) would find this a familiar approach: we choose a class of well-behaved objects, and consider the ind-completion of the full subcategory spanned by those objects.

To match with the classical definition, choose some limit presentation $\mathcal{L}^+G \simeq \lim_i G_i$ by finite-type quotients. Since we have $\mathrm{Rep}(G_i) \simeq \mathrm{Ind}(\mathrm{Rep}(G_i)^{\mathrm{fd}})$ and $\mathrm{QCoh}(\mathbb{B}\mathcal{L}^+G)^{\mathrm{fd}}$ is the Karoubi completion of $\mathrm{colim}_i \mathrm{Rep}(G_i)^{\mathrm{fd}}$ (where the connecting functors are restrictions), it tautologically follows that

$$\mathrm{Rep}(\mathcal{L}^+G) \simeq \mathrm{colim}_i \mathrm{Rep}(G_i);$$

and the essential images of the functors

$$\mathrm{Rep}(G_i)^{\mathrm{fd}} \to \mathrm{Rep}(G_i) \to \mathrm{Rep}(\mathcal{L}^+G)$$

generates $\mathrm{Rep}(\mathcal{L}^+G)$.

Now, one needs to be careful, since the category $\mathrm{Rep}(\mathcal{L}^+G)$ is *not* the derived category of the heart of its natural t-structure. In more details: the naturally defined map

$$\mathrm{Rep}(\mathcal{L}^+G) \to \mathrm{QCoh}(\mathbb{B}\mathcal{L}^+G)$$

is an equivalence for the cohomologically left-bounded part, but not otherwise. Note that in particular, the "forgetful" functor $\mathrm{Rep}(\mathcal{L}^+G) \to \mathrm{Vect}$ (which is t-exact) is *not* conservative.

Indeed, for any Artin stack (not necessarily of finite type) $\mathcal{Y}$, $\mathrm{QCoh}(\mathcal{Y})$ is both left and right t-complete [GR17, I.3, Corollary 1.5.7]. Let us now observe that $\mathrm{Rep}(\mathcal{L}^+G)$ is not left t-complete. Consider the trivial representation k. By delooping we can form $k[i] \in \mathrm{Rep}(\mathcal{L}^+G)$ for each $i \geq 0$, and consider

$$\bigoplus_{i\geq 0} k[i] \in \mathrm{Rep}(\mathcal{L}^+G);$$

the natural transformation $\mathrm{Id} \to \lim_i \tau^{\geq -i}$ coming from adjunction of truncation functors, when evaluated on this object, yields the map

$$\bigoplus_{i\geq 0} k[i] \to \prod_{i\geq 0} k[i] \in \mathrm{Rep}(\mathcal{L}^+G);$$

we claim this is not an isomorphism. Indeed, *by fiat* the object k is now compact, so we can compute

$$\mathrm{Hom}_{\mathrm{Rep}(\mathcal{L}^+G)}(k, \bigoplus_{i\geq 0} k[i]) \simeq \bigoplus_{i\geq 0} \mathrm{Hom}_{\mathrm{Rep}(\mathcal{L}^+G)}(k,k)[i]$$

and we always have

$$\mathrm{Hom}_{\mathrm{Rep}(\mathcal{L}^+G)}(k, \prod_{i\geq 0} k[i]) \simeq \prod_{i\geq 0} \mathrm{Hom}_{\mathrm{Rep}(\mathcal{L}^+G)}(k,k)[i]$$

which do not agree[6].

Remark 3.1.2. This is also the example given by Neeman in [Nee11] to demonstrate that the derived category of $\mathrm{Rep}(\mathbb{G}_a)^\heartsuit$ in characteristic $p > 0$ is not left complete. We refrain from making this comparison precise.

Now let's look back at how we chose the category $\mathrm{QCoh}(\mathbb{B}\mathcal{L}^+G)^{\mathrm{fd}}$: these are coherent sheaves on the faithfully flat cover $\mathrm{pt} \to \mathbb{B}\mathcal{L}^+G$. But there is a serious problem: this definition *depended* on the choice of this cover, while there are many other reasonable choices to make. And, because we work in the DG setting, one cannot hope to provide the homotopy data needed for the compatibility between different choices by hand. So we have to take a slightly different route to ensure homotopy coherent.

Now it's time to summarize the approach taken by Raskin and Chen when building the *renormalized* IndCoh theory. Roughly speaking, the procedure goes as follows:

1. Extend the theory of (ind-)coherent sheaves to a class of well-behaved non-finite-type *schemes*;
2. Extend the theory to colimits of objects in (1) under almost finite presentation (afp)[7] closed embeddings;
3. Extend the theory to prestacks that admit (some) faithfully flat cover by objects in (2)—we call such prestacks *weakly renormalizable*—by declaring the value of $\mathrm{IndCoh}(-)$ on it as the limit of ind-coherent sheaves over *all* such covers;
4. Prove that the result from (3) satisfies faithfully flat descent, so that we are justified to use any *particular* cover to compute its value;

[6]This is incomplete: one has to compute this hom for specific G to show they do not agree (if the hom sits in 1 degree, for instance, then no contradiction is reached). I'll fill this in later.

[7]This is the morally correct replacement of being of finite presentation in the derived world, which roughly speaking means "finitely presented in each cohomological degree". We refer readers to [Lur18, Chapter 4.1] for definition and a detailed discussion.

5. The output of (3) is not nice (e.g. compactly generated) on general prestacks. Instead, we *renormalize* this category by declaring an object to be compact if it is so when $*$-pulled back to any flat cover;

6. For the theory to be usable, of course one also needs to set up sufficiently general base change;

7. We warn that the renormalized theory no longer satisfies faithfully flat descent; nevertheless, one can show that approximations by *particular* sequences of finite-type objects (as we did above with $\mathcal{L}^+G$) behave as one expected;

8. The resulting theory is still not compatible with *factorization*, because IndCoh (even in finite-type case) do not yield sheaves of categories; more explicitly, this means for Cartesian diagrams like
$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$
the map $\mathrm{IndCoh}(Y') \otimes_{\mathrm{IndCoh}(Y)} \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X')$ is almost never an equivalence. Nevertheless, we show that if we only allow *de Rham* prestacks along the base direction (which are insensitive to nil-isomorphisms), then there is a well-behaved *crystallization* procedure that do yield *crystals* of categories that have IndCoh-like fibers (c.f. Claim 3.0.1).

Here an important caveat arises: in choosing the class of "nice-behaving schemes" in (1), it is necessary but *insufficient* to use ${}^{>-\infty}\mathrm{Sch}_{\mathrm{qcqs}}$ (the category of convergent quasi-compact, quasi-separated schemes), because the category of coherent sheaves for such schemes are in general not closed under cohomological truncation. Instead, one should only take the smaller full subcategory of objects that falls within the image of the *Noetherian approximation* functor, which is a fully faithful map
$$\mathrm{Pro}^{\mathrm{aff,ff}}({}^{>-\infty}\mathrm{Sch}_{\mathrm{aft}}) \to {}^{>-\infty}\mathrm{Sch}_{\mathrm{qcqs}}$$
where the LHS is the category of formal limits of convergent almost-finite-type schemes connected by faithfully flat affine maps. This smaller class is called *apaisant schemes*. Similarly, the word "faithfully flat" in the list above must be replaced with "faithfully descent and relatively apaisant".

Remark 3.1.3. The class of apaisant schemes is not closed under fiber products. This introduces additional technicalities when setting up the theory.

Remark 3.1.4. All variants of IndCoh theory bifurcates into the $*-$ and the $!-$ variants, which are formally dual to each other and are isomorphic to each other on laft indschemes. The $*$-theory admits $*$-pushforward for general maps, but only $*$- and !-pullbacks on nice maps, whereas the !-theory always has !-pullbacks but only $*$-pushforwards in nice cases.

3.1.5 Case of Loop Group

Once we have dealt with the arc group, it is then straightforward to extend the theory to representations of ind-(pro-)groups which are "finite type relative to $\mathcal{L}^+G$", such as the full loop group $\mathcal{L}G$ or its formal completion near $\mathcal{L}^+G$.

For instance, we can define Hecke_G as a renormalized version of $\mathrm{QCoh}(\mathrm{Gr}_G/\mathcal{L}^+G)$, in the same way as above (we require pullbacks to $\mathrm{QCoh}(\mathrm{Gr}_G)$ be coherent). Then one can show that Hecke_G inherits a convolution action, and acts on $\mathrm{Rep}(\mathcal{L}^+G)$. There is also a distinguished object

$\mathcal{O}_{\mathrm{Hecke}_G} \in \mathrm{Hecke}_G$, which is moreover an associative algebra under the convolution action. We can then define

$$\mathrm{Rep}(\mathcal{L}G) := \mathcal{O}_{\mathrm{Hecke}_G}\text{-mod}(\mathrm{Rep}(\mathcal{L}^+G)).$$

Note that the definition *depends on* the choice of the compact open subgroup $\mathcal{L}^+G$ here. Since all compact open subgroups are commensurate to each other, one can show that this choice yields equivalent $\mathrm{Rep}(\mathcal{L}G)$ for any two such choices. However, for the sake of studying *categorical* $\mathcal{L}G$-representations this is not good enough: we really need $\mathrm{Rep}(\mathcal{L}G)$ to be homotopically canonically defined. This is handled in [Ras20, Proposition 7.6.1].

Remark 3.1.6. In other words, we do not attempt to formulate $\mathrm{Rep}(\mathcal{L}G)$ by renormalizing $\mathrm{QCoh}(\mathbb{B}\mathcal{L}G)$. It is not clear to the author how feasible this path is.

Remark 3.1.7. The same discussion holds if we replace $\mathcal{L}G$ with its formal completion around $\mathcal{L}^+G$; this gives the definition of $\mathcal{L}\mathfrak{g}\text{-mod}^{\mathcal{L}^+G}$ (untwisted version). If we use a twisting (i.e. an infinitesimal line bundle) instead of the structure sheaf, we instead obtain the notion of $\hat{\mathfrak{g}}\text{-mod}_\kappa^{\mathcal{L}^+G}$.

3.2 Synopsis

The rest of this section is to give more details to the plan given above; proofs for all propositions can be found in [CF21].

Step 1

Definition 3.2.1. For $S \in {}^{>-\infty}\mathrm{Sch}_{\mathrm{qcqs}}$, define $\mathrm{Coh}(S) \subset \mathrm{QCoh}(S)$ to be the full subcategory of bounded *almost perfect* object, i.e., those objects $\mathcal{F} \in \mathrm{QCoh}(S)^+$ such that for any integer N, $\mathcal{F} \in \mathrm{QCoh}(S)^{\geq -N}$ implies $\mathcal{F}$ is a compact object in $\mathrm{QCoh}(S)^{\geq -N}$. Define $\mathrm{IndCoh}_*(S) := \mathrm{Ind}(\mathrm{Coh}(S))$.

Definition 3.2.2. $S \in {}^{>-\infty}\mathrm{Sch}_{\mathrm{qcqs}}$ is said to be *apaisant*, or $S \in \mathrm{Sch}_{\mathrm{apai}}$ if it can be written as a (co)filtered limit $S = \lim_i S_i$ under faithfully flat affine structure maps with $S_i \in {}^{>-\infty}\mathrm{Sch}_{\mathrm{aft}}$.

Definition 3.2.3. A map $f : S \to T$ in PreStk is called *relatively apaisant* if it is qcqs, schematic, of Tor dimension $\leq n$ for some integer n, and for any apaisant scheme T' over T, the fiber product $S \times_T T'$ is apaisant.

Lemma 3.2.4. *Suppose $f : S \to T$ in* PreStk *is either:*

- *qcqs, schematic, afp and of Tor dimensinon $\leq n$ for some integer n; or*
- *S can be written as a limit $S = \lim_{n\in\mathbb{N}} S_n$ under faithfully flat affine afp structure maps with $S_i \to T$ being relatively apaisant,*

then f is relatively apaisant.

Example 3.2.5. Let $H = \lim_n H_n$ be an apaisant presentation for an apaisant affine flat group scheme over a base $B \in {}^{>-\infty}\mathrm{Sch}_{\mathrm{aft}}$. Then $B \to \mathbb{B}H$ is relatively apaisant.

Step 2

Definition 3.2.6. A convergent prestack S is an *apaisant indscheme*, or $S \in \mathrm{IndSch}_{\mathrm{apai}}$, if it can be written as a filtered colimit $S \simeq \operatorname*{colim}_i S_i$ such that

- $S_i \in \mathrm{Sch}_{\mathrm{apai}}$,
- $S_i \to S_j$ are afp closed embeddings.

We define $\mathrm{IndCoh}_* : \mathrm{IndSch}_{\mathrm{apai}} \to \mathrm{DGCat}_{\mathrm{cont}}$ by left Kan extension along $\mathrm{Sch}_{\mathrm{apai}} \subset \mathrm{IndSch}_{\mathrm{apai}}$. Explicitly, we have

$$\mathrm{IndCoh}_*(S) \simeq \operatorname*{colim}_{*\text{-push}} \mathrm{IndCoh}_*(S_i) \simeq \lim_{!\text{-pull}} \mathrm{IndCoh}_*(S_i).$$

Step 3

Definition 3.2.7. A convergent prestack $S \in \mathrm{PreStk}_{\mathrm{conv}}$ is *weakly renormalizable* (which we write as $S \in \mathrm{PreStk}_{\mathrm{w.ren}}$) if there exists a faithfully flat and relatively apaisant map $T \to S$ with T being an apaisant indscheme. We refer such a map $T \to S$ as an *flat apaisant cover* for S.

Example 3.2.8. The (fppf) double quotient $\mathcal{L}^+G\backslash\mathcal{L}G/\mathcal{L}^+G$ is a weakly renormalizable prestack.

Definition 3.2.9. A map $f : S \to T$ in $\mathrm{PreStk}_{\mathrm{w.ren}}$ is *apaisant indschematic* if for any relatively apaisant map $T' \to T$ with $T' \in \mathrm{IndSch}_{\mathrm{apai}}$, the fiber product $S \times_T T'$ is contained in $\mathrm{IndSch}_{\mathrm{apai}}$.

Step 4 The following is the key fact: that *unrenormalized* IndCoh $*$-theory satisfies faithfully flat descent.

Theorem 3.2.10. *Let $f : T \to S$ be a relatively apaisant and faithfully flat map in* $\mathrm{PreStk}_{\mathrm{w.ren}}$*, then $*$-pullback induces an equivalence*

$$\mathrm{IndCoh}_*(S) \to \mathrm{Tot}_{\mathrm{semi}}(\mathrm{IndCoh}_*(T^{\times_S(\bullet+1)})).$$

Step 5

Definition 3.2.11. For $S \in \mathrm{PreStk}_{\mathrm{w.ren}}$, let $\mathrm{Coh}(S) \subset \mathrm{IndCoh}_*(S)$ be the full subcategory of objects $\mathcal{F}$ such that its $*$-pullback along any flat apaisant cover $S' \to S$ is contained in $\mathrm{Coh}(S') \subset \mathrm{IndCoh}_*(S')$. We define $\mathrm{IndCoh}_{*,\mathrm{ren}}(S) := \mathrm{Ind}(\mathrm{Coh}(S))$.

Step 6

Notation 3.2.12. For $\mathcal{C}$ a category equipped with a t-structure, let $\mathcal{C}^+$ denote the full subcategory of bounded-below objects.

The theory set up so far allows $*$-pushforward along only apaisant indschematic maps; this is not good enough for us, because in practice we are going to take some (pro)-group cohomologies. So we have to extend it a bit:

Proposition 3.2.13. *The right-lax symmetric monoidal functor*

$$(\mathrm{IndCoh}_*)^+ : (\mathrm{PreStk}_{\mathrm{w.ren}})_{\mathrm{apai.indsch}} \to \mathrm{Vect}^+\text{-mod}(\mathrm{Cat}),\ S \mapsto \mathrm{IndCoh}_*(S)^+$$

can be uniquely extended to a right-lax symmetric monoidal functor

$$(\mathrm{IndCoh}_*)^+ : \mathrm{PreStk}_{\mathrm{w.ren}} \to \mathrm{Vect}^+\text{-mod}(\mathrm{Cat})$$

satifying the following condition. Let $f : S \to T$ be a morphism in $\mathrm{PreStk}_{\mathrm{w.ren}}$ *and $q_0 : S_0 \to S$ be any flat apaisant cover. Consider the corresponding Čech nerve $S_\bullet$ and projections $q_\bullet : S_\bullet \to S$. Then the canonical natural transformation*

$$f_*^+ \to \mathrm{Tot}_{\mathrm{semi}}[((f \circ q_\bullet)_*)^+ \circ (q_\bullet^*)^+]$$

is required to be an equivalence. For any morphism $f \in \mathrm{PreStk}_{\mathrm{w.ren}}$, $(f_*)^+$ is left t-exact.

Now we are ready to spell out what kind of base change we can have. Namely, we are going to specify along which morphisms one should $*$-pull, !-pull and $*$-push. It comes in two passes: first the non-renormalzied, then the renormalized.

The previous proposition told us that the choice for non-renormalized $*$-push is *anything.* The candidate for non-renormalized $*$-pull is easy to spell out:

Definition 3.2.14. A map $f : S \to T$ in $\mathrm{PreStk}_{\mathrm{w.ren}}$ is called *$*$-pullable* if there exists a flat apaisant cover $S' \to S$ such that the composition $S' \to T$ is relatively apaisant.

Lemma 3.2.15. *(1) If $f : S \to T$ is $*$-pullable, then the functor $(f_*)^+ : \mathrm{IndCoh}_*(S)^+ \to \mathrm{IndCoh}_*(T)^+$ has a left adjoint $(f^*)^+$.*

(2) For a Cartesian diagram

$$\begin{tikzcd} S' \arrow[r, "\varphi"] \arrow[d, "\phi"] & T' \arrow[d, "g"] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

contained in $\mathrm{PreStk}_{\mathrm{w.ren}}$ *such that f and φ are $*$-pullable, the Beck-Chevalley natural transformation $(f^*)^+ \circ (g_*)^+ \to (\phi_*)^+ \circ (\varphi^*)^+$ is an isomorphism.*

The following definition is too technical to be the correct one, but it is the best we currently have. We only include it in the notes here for completeness.

Definition 3.2.16. A map $f : S \to T$ in $\mathrm{IndSch}_{\mathrm{apai}}$ is called *ind-afp* if for some apaisant presentations $S = \operatorname{colim}_{i \in \mathcal{I}} S_i$ and $T = \operatorname{colim}_{j \in \mathcal{J}} T_j$, and any index $i \in \mathcal{I}$, there exists $j \in \mathcal{J}$ such that $S_i \to S \to T$ factors through an afp (almost finitely presented) map $S_i \to T_j$.

We say $f : S \to T$ is ind-afp and *ind-proper* (resp. an *ind-afp closed embedding*) if the above maps $S_i \to T_j$ are afp and proper (resp. afp closed embeddings).

We say f is *uniformly afp*, or *uafp*, if there exists a presentation $T = \operatorname{colim}_i T_i$ of T as an apaisant indscheme such that each $T_i \times_T S$ is an apaisant *scheme* and each map $T_i \times_T S \to T_i$ is afp.

An apaisant indschematic morphism $f : S \to T$ in $\mathrm{PreStk}_{\mathrm{w.ren}}$ is *uafp* if for any relatively apaisant map $T' \to T$ with $T' \in \mathrm{IndSch}_{\mathrm{apai}}$, the map $S \times_T T' \to T'$ is a uafp morphism in $\mathrm{IndSch}_{\mathrm{apai}}$.

A uafp morphism $f : S \to T$ in $\mathrm{PreStk}_{\mathrm{w.ren}}$ is *proper* (resp. a *closed immersion*) if the uafp map $S \times_T T' \to T'$ as above is proper (resp. a closed immersion)[8].

A uafp morphism $\varphi : S' \to T'$ in $\mathrm{PreStk}_{w.ren}$ is *very afp*, or *vafp*, if it is a base-change of an afp map $f : S \to T$ in $\mathrm{Sch}_{\mathrm{apai}}$.

Now we can say what the choice is for non-renormalized !-pull *in the* $*$*-theory* (this is different from the !-pull in the !-theory defined below):

Lemma 3.2.17. *If $f : S \to T$ is uafp, then there is a canonically defined map $f^! : \mathrm{IndCoh}_*(T) \to \mathrm{IndCoh}_*(S)$, having the same adjointness property as the usual IndCoh !-pull (i.e. being right adjoint of $*$-push if proper, and left adjoint if open immersion). If f is vafp, then it preserves the bounded-below subcategory.*

Remark 3.2.18. The content of the lemma above is that the definition is *homotopy coherent*: of course, by (derived) Nagata compactification one can give *some* definition of !-pull via the adjointness property above, but then one has to show the space of choice here is homotopically contractible in the appropriate sense.

The Renormalized Version For any morphism $f : S \to T$ in $\mathrm{PreStk}_{\mathrm{w.ren}}$, the left t-exact functor $(f_*)^+ : \mathrm{IndCoh}_*(S)^+ \to \mathrm{IndCoh}_*(T)^+$ induces the following functor:

$$\mathrm{Coh}(S) \to \mathrm{IndCoh}_*(S)^+ \to \mathrm{IndCoh}_*(T)^+ \simeq \mathrm{IndCoh}_{*,\mathrm{ren}}(T)^+ \to \mathrm{IndCoh}_{*,\mathrm{ren}}(T)$$

By ind-extending, we obtain the *renormalized* $*$*-pushforward* functor

$$f_{*,\mathrm{ren}} : \mathrm{IndCoh}_{*,\mathrm{ren}}(S) \to \mathrm{IndCoh}_{*,\mathrm{ren}}(T).$$

If f is apaisant indschematic, then the above commutative diagram can be extended to the unbounded categories. By Lemma 3.2.15 and Lemma 3.2.17, we can use the above *renormalization* procedure to construct the *renormalized* $*$*-pullback functors* f^*_{ren} along any $*$-pullable morphisms and the *renormalized* !*-pullable functors* $f^!_{\mathrm{ren}}$ along any vafp morphisms.

Lemma 3.2.19. *For any Cartesian diagram*

$$\begin{tikzcd} S' \arrow[r, "\varphi"] \arrow[d, "\phi"] & T' \arrow[d, "g"] \\ S \arrow[r, "f"] & T \end{tikzcd}$$

in $\mathrm{PreStk}_{\mathrm{w.ren}}$ *such that f and φ are vafp, there is a base change isomorphism $f^!_{\mathrm{ren}} \circ g^{\mathrm{ren}}_* \simeq \phi^{\mathrm{ren}}_* \circ \varphi^!_{\mathrm{ren}}$. If g is $*$-pullable (then so is φ), the Beck-Chevalley natural transformation $\phi^*_{\mathrm{ren}} \circ f^!_{\mathrm{ren}} \to \varphi^!_{\mathrm{ren}} \circ g^*_{\mathrm{ren}}$ is an isomorphism.*

Definition 3.2.20. A morphism $f : S \to T$ be a morphism in $\mathrm{PreStk}_{\mathrm{w.ren}}$ is *ind-afp and ind-proper* (resp. *an ind-afp closed immersion*) if there exists a flat apaisant cover T_0 of T such that $S \times_T T_0 \to T_0$ is an ind-afp and ind-proper map (resp. an ind-afp closed immersion) in $\mathrm{IndSch}_{\mathrm{apai}}$.

An ind-afp and ind-proper morphism $f : S \to T$ is $(?, \mathrm{ren})$*-pullable* if f_* is left t-exact up to a shift.

[8] Recall that a uafp map in $\mathrm{IndSch}_{\mathrm{apai}}$ is schematic. Hence the notion of proper maps (resp. closed immersions) is well-defined.

Lemma 3.2.21. *(1) Let $f : S \to T$ be an ind-afp and ind-proper morphism in* $\mathrm{PreStk}_{\mathrm{w.ren}}$*, then the functor $f_{*,\mathrm{ren}}$ has a* continuous *right adjoints.*

(2) Let $f : S \to T$ be a $(?, \mathrm{ren})$-pullable map. Then the right adjoints $f^?$ and $f^?_{\mathrm{ren}}$ of f_* and $f_{*,\mathrm{ren}}$ are left t-exact up to shifts, and $f^?_{\mathrm{ren}}$ can be obtained from $f^?$ via renormalization. I.e., it can be identified with the ind-extension of

$$\mathrm{Coh}(T) \to \mathrm{IndCoh}_*(T)^+ \xrightarrow{f^?} \mathrm{IndCoh}_*(S)^+ \simeq \mathrm{IndCoh}_{*,\mathrm{ren}}(S)^+ \to \mathrm{IndCoh}_*(S)$$

such that we have a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{IndCoh}_{*,\mathrm{ren}}(T) \arrow[r, "f^?_{\mathrm{ren}}"] \arrow[d, "\mathrm{ren}"] & \mathrm{IndCoh}_{*,\mathrm{ren}}(S) \arrow[d, "\mathrm{ren}"] \\
\mathrm{IndCoh}_*(T) \arrow[r, "f^?"] & \mathrm{IndCoh}_*(S).
\end{tikzcd}$$

(3) Let

$$\begin{tikzcd}
S' \arrow[r, "\varphi"] \arrow[d, "\phi"] & T' \arrow[d, "g"] \\
S \arrow[r, "f"] & T
\end{tikzcd}$$

be a Cartesian diagram in $\mathrm{PreStk}_{\mathrm{w.ren}}$ *such that $f^?_{\mathrm{ren}}$ and $\varphi^?_{\mathrm{ren}}$ are well-defined. Then the Beck-Chevalley natural transformation $\phi_{*,\mathrm{ren}} \circ \varphi^?_{\mathrm{ren}} \to f^?_{\mathrm{ren}} \circ g_{*,\mathrm{ren}}$ is an isomorphism.*

If g is $*$-pullable (hence so is ϕ), then the Beck-Chevalley natural transformation $\phi^*_{\mathrm{ren}} \circ f^?_{\mathrm{ren}} \to \varphi^?_{\mathrm{ren}} \circ g^*_{\mathrm{ren}}$ is also an isomorphism.

If g and ϕ are vafp, then the Beck-Chevalley natural transformation $\phi^!_{\mathrm{ren}} \circ f^?_{\mathrm{ren}} \to \varphi^?_{\mathrm{ren}} \circ g^!_{\mathrm{ren}}$ is also an isomorphism.

Variant 3.2.22. For $S \in \mathrm{PreStk}_{\mathrm{w.ren}}$, the category $\mathrm{IndCoh}_{*,\mathrm{ren}}(S)$ is compactly generated hence dualizable. We define $\mathrm{IndCoh}^{!,\mathrm{ren}}(S) := \mathrm{IndCoh}_{*,\mathrm{ren}}(S)^\vee \simeq \mathrm{Ind}(\mathrm{Coh}(S)^{\mathrm{op}})$. For $f : S \to T$, we define the following functors:

- The *renormalized* !*-pullback functor for* $\mathrm{IndCoh}^!$*-theory*

  $$f^{!,\mathrm{ren}} : \mathrm{IndCoh}^{!,\mathrm{ren}}(T) \to \mathrm{IndCoh}^{!,\mathrm{ren}}(S)$$

  is defined as the dual of $f_{*,\mathrm{ren}}$.

- When f is $*$-pullable, the *renormalized* ?*-pushforward functor for* $\mathrm{IndCoh}^!$*-theory*

  $$f^{\mathrm{ren}}_? : \mathrm{IndCoh}^{!,\mathrm{ren}}(S) \to \mathrm{IndCoh}^{!,\mathrm{ren}}(T),$$

  is defined as the dual of f^*_{ren} or equivalently as the *continuous* right adjoint of $f^{!,\mathrm{ren}}$.

  We use ?*-pushable* as a synonym for $*$*-pullable*.

- When f is vafp, the *renormalized* $*$*-pushforward functor for* $\mathrm{IndCoh}^!$*-theory*

  $$f^{\mathrm{ren}}_* : \mathrm{IndCoh}^{!,\mathrm{ren}}(S) \to \mathrm{IndCoh}^{!,\mathrm{ren}}(T)$$

  is defined as the dual of f_{ren}.

- When f is $(?,\mathrm{ren})$-pullable, the *renormalized !-pushforward functor for* $\mathrm{IndCoh}^!$*-theory*

$$f_!^{\mathrm{ren}} : \mathrm{IndCoh}^{!,\mathrm{ren}}(S) \to \mathrm{IndCoh}^{!,\mathrm{ren}}(T),$$

is defined as the dual of $f^?_{\mathrm{ren}}$ or equivalently as the left adjoint of $f^{!,\mathrm{ren}}$.

We use $(!,\mathrm{ren})$-*pushable* as a synonym for $(?,\mathrm{ren})$-pullable.

Step 7

Proposition 3.2.23. *Suppose $S \in \mathrm{PreStk}_{\mathrm{w.ren}}$ can be written as a filtered colimit $S \simeq \operatorname*{colim}_{\alpha\in I} S^\alpha$ such that:*

- *The connecting maps $\iota_{\alpha\to\beta} : S^\alpha \to S^\beta$ and $\iota_\alpha : S^\alpha \to S$ are apaisant indschematic;*
- *There exists a flat apaisant cover $S_0 \to S$ such that the map $S^\alpha \times_S S_0 \to S^\beta \times_S S_0$ is an ind-afp closed immersion in $\mathrm{IndSch}_{\mathrm{apai}}$.*

Then the functors

$$\operatorname*{colim}_{*\text{-push}} \mathrm{IndCoh}_*(S^\alpha) \to \mathrm{IndCoh}_*(S) \qquad \operatorname*{colim}_{*\text{-push}} \mathrm{IndCoh}_{*,\mathrm{ren}}(S^\alpha) \to \mathrm{IndCoh}_{*,\mathrm{ren}}(S)$$

are equivalences.

Proposition 3.2.24. *Let $B \in {}^{>-\infty}\mathrm{Sch}_{\mathrm{aft}}$ be a base scheme, $T = \lim_{n\in\mathbb{N}} T^n$ be an apaisant presentation for an apaisant scheme over B, and $H = \lim_{n\in\mathbb{N}} H^n$ be an apaisant presentation for an apaisant affine flat group scheme over B. Suppose we have compatible actions of H^n on T^n relative to B. Then:*

(0) The fppf quotient stack T^n/H^n is a QCA stack[9]*.*

(1) The fppf quotient stack T/H is weakly renormalizable and $T \to T/H$ is a flat apaisant cover.

(2) The maps $T/H \to T^n/H^n$ and $T^n/H^n \to T^m/H^m$ are $*$-pullable.

(3) The functor

$$\operatorname*{colim}_{*\text{-pull}} \mathrm{IndCoh}(T^n/H^n) \to \mathrm{IndCoh}_{*,\mathrm{ren}}(T/H)$$

is an equivalence. In particular, the functor

$$\mathrm{IndCoh}_{*,\mathrm{ren}}(T/H) \to \lim_{*\text{-push}} \mathrm{IndCoh}(T^n/H^n)$$

is also an equivalence.

Definition 3.2.25. Let $B \in {}^{>-\infty}\mathrm{AffSch}_{\mathrm{aft}}$ be an affine base scheme. A weakly renormalizable prestack S over B is *of quotient type* if it can be written as $S \simeq \operatorname*{colim}_{\alpha\in I} S^\alpha$ as in Proposition 3.2.23 such that each S^α is of the form $S^\alpha \simeq T/H$ as in Proposition 3.2.24 with $T \to B$ of bounded Tor dimension.

[9] This is a condition introduced by Drinfeld and Gaitsgory in [DG13]. A morphism being QCA implies that QCoh $*$-pushforward is nicely behaved: it is colimit-preserving, satisfies base-change with respect to any $*$-pullback, and satisfies projection formula. We warn that all these fail for general maps between prestacks.

Definition 3.2.26. For a laft prestack $B \in {}^{>-\infty}\mathrm{AffSch}_{\mathrm{aft}}$. A convergent prestack Y_B over B is *of quotient type* if for *any* test scheme $s : S \to B$ with $S \in {}^{>-\infty}\mathrm{AffSch}_{\mathrm{aft}}$, the fiber product $Y_s := Y_B \times_B S$ is of quotient type over S.

Example 3.2.27. $[\mathcal{L}^+G\backslash\mathcal{L}G/\mathcal{L}^+G]_{\mathrm{Ran}_{\mathrm{dR}}} \to \mathrm{Ran}_{\mathrm{dR}}$ is of quotient type over $\mathrm{Ran}_{\mathrm{dR}}$.

Remark 3.2.28. If Y_B is of quotient type over $B \in {}^{>-\infty}\mathrm{Sch}_{\mathrm{aft}}$, then Y_B is weakly renormalizable.

3.3 Categorical Representations

The above finishes the development of the theory at a point; it remains to treat factorization. Before we do that, however, we pause to say a few words about some natural consequences.

Let's start with G a finite-type affine group scheme. In such case, define

$$G\text{-ModCat} := (\mathrm{QCoh}(G), \star)\text{-mod}(\mathrm{DGCat})$$

here $\star$ indicates that we use the convolution monoidal structure on $\mathrm{QCoh}(G)$. The trivial module functor (restriction along $\mathrm{QCoh}(G) \to \mathrm{Vect}$ given by global section)

$$\mathrm{triv} : \mathrm{DGCat} \to G\text{-ModCat}$$

admits a left adjoint

$$\mathrm{coinv}_G : \mathcal{C} \mapsto \mathrm{Vect} \otimes_{\mathrm{QCoh}(G)} \mathcal{C}$$

and a right adjoint

$$\mathrm{inv}_G : \mathcal{C} \mapsto \mathrm{Hom}_{\mathrm{QCoh}(G)\text{-mod}}(\mathrm{Vect}, \mathcal{C})$$

The rigidity of $\mathrm{QCoh}(G)$ then tells us that Vect is self-dual as a $\mathrm{QCoh}(G)$-module, from which it follows that the natural map

$$\mathrm{coinv}_G(\mathcal{C}) \to \mathrm{inv}_G(\mathcal{C})$$

is an equivalence. Moreover, note that inv_G factors through the category

$$\mathrm{Hom}_{\mathrm{QCoh}(G)\text{-mod}}(\mathrm{Vect}, \mathrm{Vect})\text{-mod}(\mathrm{DGCat}) \simeq \mathrm{Rep}(G)\text{-mod}(\mathrm{DGCat});$$

the same rigidity property shows that the functor

$$G\text{-ModCat} \to \mathrm{Rep}(G)\text{-mod}(\mathrm{DGCat})$$

is an equivalence, with the inverse given by the de-equivariantization functor $\mathcal{C} \mapsto \mathrm{Vect} \otimes_{\mathrm{Rep}(G)} \mathcal{C}$.

Remark 3.3.1. Actually the last statement is stronger than the first one: indeed, note that the first map is part of a Koszul adjunction

$$\bullet \otimes_{\mathrm{Rep}(G)} \mathrm{Vect} : \mathrm{Rep}(G)\text{-mod}(\mathrm{DGCat}) \rightleftarrows \mathrm{QCoh}(G)\text{-mod} : \mathrm{Hom}_{\mathrm{QCoh}(G)}(\mathrm{Vect}, \bullet)$$

tracing through the definition, one sees that the map $\mathrm{coinv}_G \to \mathrm{inv}_G$ is just the unit of this adjunction followed by the forgetful functor to DGCat.

Now we ask, what is the counterpart of this for $\mathcal{L}^+G$? Following the idea from 1-categorical level down, we propose

$$\mathcal{L}^+G\text{-ModCat} := \mathrm{colim}_i\, G_i\text{-ModCat}$$

where the connecting functors are given by restrictions. In this case, there is an induced functor

$$\mathrm{oblv}: \mathcal{L}^+G\text{-ModCat} \to \mathrm{QCoh}(\mathcal{L}^+G)\text{-mod}(\mathrm{DGCat}) \to \mathrm{DGCat}$$

but this functor is *not* conservative (it fails at the first step).[10] In contrast, there is another functor

$$\mathrm{inv}_{\mathcal{L}^+G}: \mathcal{L}^+G\text{-ModCat} \to \mathrm{Rep}(\mathcal{L}^+G)\text{-mod}(\mathrm{DGCat}) \to \mathrm{DGCat}$$

which is conservative, and in fact the first step is an equivalence. This functor is both the left and the right adjoint of the functor $\mathrm{triv}: \mathrm{DGCat} \to \mathrm{Rep}(\mathcal{L}^+G)\text{-mod}(\mathrm{DGCat})$, given by e.g. restricting along $\mathrm{Rep}(\mathrm{pt}) \simeq \mathrm{Vect} \to \mathrm{Rep}(\mathcal{L}^+G)$. In other words, the de-equivariantization equivalence fails, we still have an invariant-equals-coinvariant statement.

Remark 3.3.2. There is in fact a (non-conservative) functor

$$\Psi: \mathrm{Rep}(\mathcal{L}^+G)\text{-mod}(\mathrm{DGCat}) \to \mathrm{QCoh}(\mathcal{L}^+G)\text{-mod}(\mathrm{DGCat})$$

which admits fully faithful left and right adjoints. This is reminiscent of the relationship between IndCoh and QCoh on (not necessarily smooth) schemes.

Now what happens with $\mathcal{L}G$? Certainly $\mathrm{QCoh}(\mathcal{L}G)$-mod isn't a good answer, but there are two choices:

- Following the strategy from 1-categorical level down, we could require that taking $\mathcal{L}^+G$-weak invariant should be monadic. In this setting this simply amounts to *defining*

  $$\mathcal{L}G\text{-ModCat} := \mathrm{Hecke}_G\text{-mod}(\mathrm{DGCat})$$

  and calling the resulting forgetful functor $\mathcal{L}G\text{-ModCat} \to \mathrm{DGCat}$ by the name $\mathrm{inv}_{\mathcal{L}^+G}$;
- Or we could have declaring it should have been $\mathrm{Rep}(\mathcal{L}G)\text{-mod}(\mathrm{DGCat})$.

The first approach is what we'll take; we shall see in a minute why. There is a natural functor $\mathrm{triv}: \mathrm{DGCat} \to \mathcal{L}G\text{-ModCat}$ (coming from functoriality of the definition of Hecke_G), and we define

$$\mathrm{inv}_{\mathcal{L}G}, \mathrm{coinv}_{\mathcal{L}G}: \mathcal{L}G\text{-ModCat} \to \mathrm{DGCat}$$

to be its left and right adjoints respectively. These can be computed explicitly as

$$\mathrm{inv}_{\mathcal{L}G}(\mathcal{C}) \simeq \mathrm{Hom}_{\mathrm{Hecke}_G}(\mathrm{Rep}(\mathcal{L}^+G), \mathrm{inv}_{\mathcal{L}^+G}(\mathcal{C})) \quad \mathrm{coinv}_{\mathcal{L}G}(\mathcal{C}) \simeq \mathrm{Rep}(\mathcal{L}^+G) \otimes_{\mathrm{Hecke}_G} \mathrm{inv}_{\mathcal{L}^+G}(\mathcal{C}).$$

Observe that $\mathrm{inv}_{\mathcal{L}G}$ factors through the category $\mathrm{Rep}(\mathcal{L}G)\text{-mod}(\mathrm{DGCat})$; however $\mathrm{inv}_{\mathcal{L}G}$ is *not* conservative. We think of this as indicating that the map $\mathcal{L}G\text{-ModCat} \to \mathrm{Rep}(\mathcal{L}G)\text{-mod}(\mathrm{DGCat})$ is lossy, so it's better to take the first choice.

Another issue arises: for $\mathcal{L}G$, invariance and coinvariance *do not* agree. The functor $\mathrm{coinv}_{\mathcal{L}G}$ turns out to be corepresentable by an object of $\mathcal{L}G$-ModCat, which we'll refer to as the *modular character* and denote by χ_G. This object is obtained by inducing $\mathrm{triv}(\mathrm{Vect}) \in \mathcal{L}G\text{-ModCat}$ along the *Serre automorphism* of the rigid monoidal category Hecke_G.

[10] Some details: the functor factors through the map $\mathrm{Rep}(\mathcal{L}^+G)\text{-mod}(\mathrm{DGCat}) \to \mathrm{QCoh}(\mathbb{B}\mathcal{L}^+G)\text{-mod}(\mathrm{DGCat})$ by inducing along the symmetric monoidal map $\mathrm{Rep}(\mathcal{L}^+G) \to \mathrm{QCoh}(\mathbb{B}\mathcal{L}^+G)$ followed by the Koszul adjunction map. We claim the latter is not conservative: this follows from combining Proposition 6.1.2 and Theorem 2.2.3 of [Gai15a].

Remark 3.3.3. Recall that if $\mathcal{A}$ is a Frobenius category (implied by rigidity), then there exists an automorphism σ of it characterized by the isomorphism

$$\mathrm{Hom}(1_{\mathcal{A}}, A \otimes B) \simeq \mathrm{Hom}(1_{\mathcal{A}}, B \otimes \sigma(A))$$

for all $A, B \in \mathcal{A}$.

Under the symmetric monoidal structure on $\mathcal{L}G$-ModCat, the modular character is invertible, with the inverse χ_G^{-1} given by pushing forward along the inverse of the Serre automorphism. It follows from definition that we have

$$\mathrm{coinv}_{\mathcal{L}G} \simeq \mathrm{inv}_{\mathcal{L}G}(\bullet \otimes \chi_G^{-1}).$$

Moreover, as a $\mathrm{Rep}(\mathcal{L}^+G)$-module, χ_G is canonically isomorphic to triv(Vect). With more effort, one can show that:

Proposition 3.3.4. *This module is via a group morphism $\mathcal{L}G \to \mathbb{B}\mathbb{G}_m$, i.e. corresponding to a central extension of $\mathcal{L}G$, which is, in fact, the* Tate *central extension of $\mathcal{L}G$.*

We remark that this observation has immediate representation-theoretic consequences. Indeed, one can show that $\mathrm{DMod}_\kappa(\mathcal{L}G)$ belongs to $\mathcal{L}G$-ModCat, and the definition

$$\hat{\mathfrak{g}}\text{-mod}_\kappa \coloneqq \mathrm{inv}_{\mathcal{L}G}(\mathrm{DMod}_\kappa(\mathcal{L}G))$$

turns out to recover the *renormalized* derived category of Kac-Moody representations introduced in [FG09]. The existence of the modular character, however, implies that this category is *not* self-dual as an element of $\mathcal{L}G$-ModCat, but instead gets twisted by the Tate line bundle. This is the source of the *critical shift* ubiquitous within affine representation theory. Similarly, as explained during the talk, this accounts for the critical shift that appeared when taking the quantum BRST reduction.

3.4 Crystallization

Now we explain how to make IndCoh factorize. This is one of the key new technical advancements of our main paper.

Definition 3.4.1. Let Y_B be of quotient type over a laft prestack B. For any laft prestack $T \to B$ over B, we define

$$\mathbf{\Gamma}(t : T \to B, \mathcal{I}nd\mathcal{C}oh_{*,\mathrm{ren}}(Y)) \coloneqq \lim_{!\text{-pull}, S \to T} \mathrm{IndCoh}_{*,\mathrm{ren}}(Y_s),$$

where the limit is taken for any test scheme $S \to T$ with $S \in {}^{>-\infty}\mathrm{AffSch}_{\mathrm{aft}}$, and s is the composition $S \to T \to B$ (the !-pullback functors are well-defined because each $Y_{s'} \to Y_s$ is vafp). When there is no ambiguity, we also write $\mathbf{\Gamma}(T, \mathcal{I}nd\mathcal{C}oh_{*,\mathrm{ren}}(Y))$.

By construction, $\mathbf{\Gamma}(T, \mathcal{I}nd\mathcal{C}oh_{*,\mathrm{ren}}(Y))$ has a module structure for the symmetric monoidal category $\mathrm{IndCoh}(T) \simeq \lim_{!\text{-pull}, S \to T} \mathrm{IndCoh}(S)$ which is functorial in T.

The main statement is the following:

Proposition 3.4.2. *Let $Y_{B_{\mathrm{dR}}}$ be of quotient type over a laft de-Rham prestack B_{dR}. Then*

$$[T \in ({}^{>-\infty}\mathrm{AffSch}_{\mathrm{aft}})_{/B}] \mapsto [\mathbf{\Gamma}(T_{\mathrm{dR}}, \mathcal{I}nd\mathcal{C}oh_{*,\mathrm{ren}}(Y)) \in \mathrm{DMod}(T)\text{-mod}]$$

defines a crystal of categories on B_{dR}. In other words, for any $T' \to T$, the functor

$$\mathbf{\Gamma}(T_{\mathrm{dR}}, \mathcal{I}nd\mathcal{C}oh_{*,\mathrm{ren}}(Y)) \otimes_{\mathrm{DMod}(T)} \mathrm{DMod}(T') \to \mathbf{\Gamma}(T'_{\mathrm{dR}}, \mathcal{I}nd\mathcal{C}oh_{*,\mathrm{ren}}(Y))$$

is an equivalence.

A few remarks:

- Note that we are only specifying the sections along de Rham base changes; in other words, for a general map $t : T \to B_{\mathrm{dR}}$ from some scheme T, we have no control over the corresponding section of $\mathcal{I}nd\mathcal{C}oh_{*,\mathrm{ren}}(Y)$;
- Even for a de Rham base change $t_{\mathrm{dR}} : T_{\mathrm{dR}} \to B_{\mathrm{dR}}$, it is *not* always the case that the resulting category is simply $\mathrm{IndCoh}_{*,\mathrm{ren}}(Y_{t_{\mathrm{dR}}})$. Rather, the section is the limit over all base change along $S \to T_{\mathrm{dR}} \to B_{\mathrm{dR}}$ to $S \in {}^{>-\infty}\mathrm{AffSch}_{\mathrm{aft}}$. By general theory, one can restrict to those S that is *nil-isomorphic*[11] to T_{dR}. In other words, we fix a basic "geometric shape" (T_{dR}) and take a limit over approximations by all schemes that have this underlying shape;
- However, in some cases the section can be shown to cincide with $\mathrm{IndCoh}_{*,\mathrm{ren}}(Y_{t_{\mathrm{dR}}})$: for instance, when the fibers of Y_{dR} are all almost of finite-type, or when T is simply a point. In general, the result is sensitive to the specific geometry of Y and T;
- The fact that base change now works boils down to the following geometric fact: suppose we have $X \to Y$ a map of laft prestacks, then we have

  $$X_{\mathrm{dR}} \simeq \mathrm{colim}_{S \to X_{\mathrm{dR}}, V \to Y_{\mathrm{dR}}} S \times_{Y_{\mathrm{dR}}} V$$

  where S, V ranges over ${}^{>-\infty}\mathrm{AffSch}_{\mathrm{aft}}$.

Variant 3.4.3. One can show that $\mathcal{I}nd\mathcal{C}oh_{*,\mathrm{ren}}(Y)$ is a dualizable object in the symmetric monoidal category $\mathrm{ShvCat}(B_{\mathrm{dR}})$, and the subsections of its dual $[\mathcal{I}nd\mathcal{C}oh_{*,\mathrm{ren}}(Y)]^\vee$ are given by

$$\mathbf{\Gamma}(T_{\mathrm{dR}}, [\mathcal{I}nd\mathcal{C}oh_{*,\mathrm{ren}}(Y)]^\vee) \simeq \mathbf{\Gamma}(T_{\mathrm{dR}}, \mathcal{I}nd\mathcal{C}oh_{*,\mathrm{ren}}(Y))^\vee \simeq \lim_{!\text{-pull}, S \to T_{\mathrm{dR}}} \mathrm{IndCoh}^{!,\mathrm{ren}}(Y_s).$$

This identification is functorial in T_{dR}, so we can use the notation

$$\mathcal{I}nd\mathcal{C}oh^{!,\mathrm{ren}}(Y) := [\mathcal{I}nd\mathcal{C}oh_{*,\mathrm{ren}}(Y)]^\vee.$$

Let $Y_{\mathrm{Ran}_{\mathrm{dR}}} \to \mathrm{Ran}_{\mathrm{dR}}$ be a factorization space over $\mathrm{Ran}_{\mathrm{dR}}$ such that $Y_{\mathrm{Ran}_{\mathrm{dR}}}$ is of quotient type over $\mathrm{Ran}_{\mathrm{dR}}$. One can show that there is a factorization structure on the crystals of categories $\mathcal{I}nd\mathcal{C}oh_{*,\mathrm{ren}}(Y)$ and $\mathcal{I}nd\mathcal{C}oh^{!,\mathrm{ren}}(Y)$ over $\mathrm{Ran}_{\mathrm{dR}}$, which are dual to each other in the symmetric monoidal category of factorization categories over $\mathrm{Ran}_{\mathrm{dR}}$. By definition, for any closed point $x \in X$, the fibers of these factorization categories at x are $\mathrm{IndCoh}_{*,\mathrm{ren}}(Y_x)$ and $\mathrm{IndCoh}^{!,\mathrm{ren}}(Y_x)$.

With sufficient bookkeeping, one can extend this to the unital setting: if $Y_{\mathrm{Ran}_{\mathrm{dR}}}$ has a corr-unital factorization space structure, then the resulting category has a unital structure as well.

[11]A map between prestacks is a nil-isomorphism if it induces an isomorphism on the underlying classical reduced prestacks.

4 Talk 4: Local-Global Correspondence

Let our global curve X now be smooth and *projective*; in reality we only need for $\mathbb{P}^1$. Our final goal is to construct the following commutative diagram

$$\begin{tikzcd} \hat{\mathfrak{g}}\text{-mod}^I_\kappa \simeq \mathrm{Vac}\text{-mod}^{\mathrm{Fact}}_{\mathrm{un}}(\hat{\mathfrak{g}}\text{-mod}^I_\kappa) \arrow[r, "C^{\infty/2}_*(\mathcal{L}\mathfrak{n},-)"] \arrow[d, "\mathrm{Loc}"] & \Omega\text{-mod}^{\mathrm{Fact}}_{\mathrm{un}}(\mathrm{KL}_{\kappa'}(T)) \arrow[d, "\mathrm{Loc}_\Omega"] \\ \mathrm{DMod}_\kappa(\mathrm{Bun}_G^{B\text{-level}}) \arrow[r, "\mathrm{CT}_*"] & \mathrm{DMod}_{\kappa'}(\mathrm{Bun}_T) \end{tikzcd}$$

where:

- $\mathrm{Bun}_G^{B\text{-level}} \simeq \mathrm{Bun}_G \times_{\mathbb{B}G} \mathbb{B}B$ is the moduli space of G-bundles on $\mathbb{P}^1$ along with a B-reduction at 0;
- the top horizontal arrow is the *factorizable* semi-infinite cohomology functor, rather, the induced functor between factorization modules (recall that Ω is the semi-infinite cohomology of the vacuum);
- the bottom arrow is, up to shifting by a Tate line bundle, the !-pull-$*$-push along the correspondence diagram $\mathrm{Bun}_G \leftarrow \mathrm{Bun}_B \rightarrow \mathrm{Bun}_T$. This is called the geometric constant term functor in Langlands;
- We'll explain the vertical arrows below.

Let's first understand why this is a key diagram. Recall from the main talk that in matching up $\hat{\mathfrak{g}}\text{-mod}^I_\kappa$ with Ω-factorization modules, the hard part is to check that factorizable semi-infinite cohomology (along with FLE for tori) send *standard* objects (which are not Verma modules but duals of type 1 Wakimoto modules) to standard objects in the category of Ω-factorization modules. The latter are defined as !-extensions from substrata, so are characterized by their $*$-fibers.

These $*$-fibers are hard to compute from definition. However, one can proceed as follows:

1. First, for any $\check{\mu}$ in the weight lattice we can extend the Ω-factorization module $C^{\infty/2}_*(\mathcal{M})$, which lives on the configuration space $\mathrm{Conf}(\mathbb{A}^1)$ of weight-lattice-colored divisors on $\mathbb{A}^1$, to a factorization module $\mathcal{F}$ on $\mathbb{P}^1$ via $*$-extension along $\mathrm{add}_{\check{\mu}\cdot\infty} : \mathrm{Conf}(\mathbb{A}^1) \to \mathrm{Conf}(\mathbb{P}^1)$; this tautologically amounts to putting a *costandard object* $\nabla^{\check{\mu}}$ (type w_0 Wakimoto) of highest weight $\check{\mu}$ at $\infty \in \mathbb{P}^1$;
2. Suppose we are trying to compute $*$-fiber at $\check{\lambda}\cdot 0$. Recall that components of $\mathrm{Conf}(\mathbb{P}^1)$ is marked by the total weights of the divisors; we can choose $\check{\mu}$ such that the twisting gerbe vanishes on the component of $\mathrm{Conf}(\mathbb{P}^1)$ corresponding to $\check{\mu} + \check{\lambda}$ (in practice this amounts to $\check{\mu} + \check{\lambda} = 2\check{\rho}$);
3. Using contraction principle one can then match the $*$-fiber with the Ω-conformal block $\langle \mathcal{M}, \nabla^{\check{\mu}} \rangle_\Omega$, defined as the pairing between $\mathrm{Loc}_\Omega(\mathcal{F})$ and the delta-sheaf supported at 0. *By the diagram above*, this is the result of pairing $\mathrm{Loc}(\mathcal{M}, \nabla^{\check{\mu}})$ with $\mathrm{Eis}_*(\delta_0)$, here Eis_* is[12] the dual functor of CT_*, given by !-pull-$*$-push along $\mathrm{Bun}_T \leftarrow \mathrm{Bun}_B \rightarrow \mathrm{Bun}_G$. The latter finally becomes the *usual* conformal block $\langle \mathcal{M}, \nabla^{\check{\mu}} \rangle$ which is easier to compute.

[12] One should be careful with the self-duality of $\mathrm{DMod}(\mathrm{Bun}_G)$, but let's ignore that for now.

Remark 4.0.1. On the Whittaker side of local Langlands we have a similar diagram, where we replace Loc with the functor of Poincare series (which maps from the local unramified Whittaker category to $\mathrm{DMod}_\kappa(\mathrm{Bun}_G)$). In this case, the diagram categories a (surely well known) number-theoretic fact expressing constant term of Poincare series in terms of local factors.

For the remainder of this talk we'll instead use the following diagram, whose proof is exactly the same as the one above:

$$\begin{tikzcd}
\hat{\mathfrak{g}}\text{-mod}_\kappa^{\mathcal{L}^+G} \arrow[r, "C_*^{\infty/2}(\mathcal{L}\mathfrak{n},-)"] \arrow[d, "\mathrm{Loc}"] & \Omega\text{-mod}_{\mathrm{un}}^{\mathrm{Fact}}(\mathrm{KL}_{\kappa'}(T)) \arrow[d, "\mathrm{Loc}_\Omega"] \\
\mathrm{DMod}_\kappa(\mathrm{Bun}_G) \arrow[r, "\mathrm{CT}_*"] & \mathrm{DMod}_{\kappa'}(\mathrm{Bun}_T)
\end{tikzcd}$$

4.1 Chiral Localization

Let us review the "usual" localization introduced by Beilinson and Drinfeld, which is a functor

$$\mathrm{Loc} : \hat{\mathfrak{g}}\text{-mod}_\kappa^{\mathcal{L}^+G} \to \mathrm{DMod}_\kappa(\mathrm{Bun}_G);$$

Recall that $\mathrm{Bun}_G(\mathbb{P}^1)$ has a Weil uniformization

$$\mathrm{Bun}_G(\mathbb{P}^1) \simeq \mathcal{L}^+G\backslash\mathcal{L}G/G[t^{-1}]$$

and the "thick" affine Grassmannian $\mathcal{L}G/G[t^{-1}]$ is acted on by $\mathcal{L}G$ from left, so we have a localization functor $\mathcal{L}\mathfrak{g}\text{-mod} \to \mathrm{DMod}(\mathcal{L}G/G[t^{-1}])$. Turning on twisting we obtain $\hat{\mathfrak{g}}\text{-mod}_\kappa \to \mathrm{DMod}_\kappa(\mathcal{L}G/G[t^{-1}])$, and the $G(O)$-equivariant version is Loc.

Remark 4.1.1. Note that we are being a bit careless here, since we did not account for the renormalization issue much emphasized above.

From this description it is clear that this localization functor is an affine analogue of the Beilinson-Bernstein localization. Here's a different way to look at it:

Proposition 4.1.2. *The !-fiber of* $\mathrm{Loc}(M)$ *at the trivial* G*-bundle is the conformal block of* M *on* $\mathbb{P}^1$*, seen as a chiral module for the vacuum placed at* 0*. More generally, the !-fiber at a different bundle* $\mathcal{P}_G$ *is the conformal block of* M *seen as a chiral module of the* $\mathcal{P}_G$*-twisted vacuum*[13]*.*

In other words, $\mathrm{Loc}(M)$ can be seen as remembering the flat connection on the vector bundle of (twisted) conformal blocks on Bun_G. We now generalize this latter perspective.

Recall that a key feature of conformal block is that it doesn't change if we insert vacuum itself (as a chiral module) at some other point, i.e.

$$\langle M, \mathrm{Vac}\rangle \simeq \langle M, \mathrm{Vac}, \mathrm{Vac}\rangle \ldots\ldots$$

if all modules are assumed to be at different points. Now, let A be an unital chiral algebra and M a chiral module of A (which is also a chiral module over the vacuum, by restriction), then we have maps of conformal blocks

$$\langle M\rangle \simeq \langle M, \mathrm{Vac}\rangle \to \langle M, A\rangle \to \langle M, A, A\rangle \to \ldots;$$

[13] By which we mean the following: recall that the vacuum is the chiral envelope of the Lie-$*$ algebra $\mathfrak{g}\otimes D_X$, which is the induced D-module from $\otimes\mathcal{O}_X$. To form the twisted version, replace the last object with $\mathfrak{g}\times^G\mathcal{P}_G$.

and let us define $\langle M\rangle_A$ ("the A-conformal block") as the colimit of these conformal blocks over all possible places of insertions. More precisely, there turns out to be a D-module on Ran_0 (the set of finite subsets containing 0) whose !-fiber at $(0, x_1, \dots, x_k)$ is $\langle M, A_{\text{at } x_1}, \dots, A_{\text{at } x_k}\rangle$; our $\langle M\rangle_A$ is then the global section of this D-module.

Our generalized version

$$\mathrm{Loc}_A : A\text{-mod}^{\mathrm{Fact}}_{\mathrm{un}}(\hat{\mathfrak{g}}\text{-mod}_\kappa^{\mathcal{L}^+G}) \to \mathrm{DMod}_\kappa(\mathrm{Bun}_G)$$

will have the following property: the !-fiber of $\mathrm{Loc}_A(M)$ at the trivial bundle is precisely $\langle M\rangle_A$. The functor Loc introduced above turns out to coincide with $\mathrm{Loc}_{\mathrm{Vac}}$, as expected.

The construction of Loc_A (due to Nick Rozenblyum) factors as

$$A\text{-mod}^{\mathrm{Fact}}_{\mathrm{un}}(\hat{\mathfrak{g}}\text{-mod}_\kappa^{\mathcal{L}^+G}) \xrightarrow{\mathrm{oblv}} \mathrm{KL}_\kappa(G)_{\mathrm{Ran}} \to \mathrm{DMod}_\kappa(\mathrm{Bun}_G)$$

so we'll explain the second half. It is notationally easier to describe the untwisted version $\mathrm{KL}(G)_{\mathrm{Ran}} \to \mathrm{DMod}(\mathrm{Bun}_G)$.

Recall that an element of $\mathrm{KL}(G)$ can be described as an element of $\mathrm{Rep}(\mathcal{L}^+G)$ along with equivariance with respect to the convolution action by the structure sheaf of the infinitesimal Hecke stack $\mathrm{Hecke}_{\mathrm{inf},x} := \mathcal{L}^+G\backslash \mathcal{L}G^\wedge_{\mathcal{L}^+G}/\mathcal{L}^+G$. For a fixed point x, evaluating at the formal disc around x gives a map $\mathrm{Bun}_G \to \mathbb{B}\mathcal{L}^+G$, so we can form the product

$$\mathrm{Hecke}_{\mathrm{inf,glob},x} := \mathrm{Bun}_G \times_{\mathbb{B}\mathcal{L}^+G} \mathrm{Hecke}_{\mathrm{inf}},$$

whose S-points classify the following data: two G-bundles on $S \times X$, an isomorphism α of them away from the graph, an isomorphism β of them as G-bundles on $S^{\mathrm{cl,red}} \times X$ (recall $S^{\mathrm{cl,red}}$ is the classical and reduced subscheme of S), and a compatibility between α and β. The via pulling back, we have a functor

$$\mathrm{KL}(G)_x \simeq \mathcal{O}_{\mathrm{Hecke}_{\mathrm{inf}}}\text{-mod}(\mathrm{Rep}(\mathcal{L}^+G)) \to \mathcal{O}_{\mathrm{Hecke}_{\mathrm{inf,glob},x}}\text{-mod}(\mathrm{IndCoh}(\mathrm{Bun}_G)).$$

The latter can be seen as an intermediate step between $\mathrm{IndCoh}(\mathrm{Bun}_G)$ and $\mathrm{DMod}(\mathrm{Bun}_G)$: indeed, $\mathrm{IndCoh}(\mathrm{Bun}_G)$ is tautologically the category of modules over the trivial groupoid $\mathrm{Bun}_G \rightrightarrows \mathrm{Bun}_G$, and $\mathrm{DMod}(\mathrm{Bun}_G)$ is (by definition) modules over the tangent groupoid (i.e. formal completion along the diagonal)

$$\mathrm{Hecke}_{\mathrm{inf,glob},X} := (\mathrm{Bun}_G \times \mathrm{Bun}_G)^\wedge_{\mathrm{Bun}_G} \rightrightarrows \mathrm{Bun}_G;$$

We have natural maps

$$\mathrm{Bun}_G \to \mathrm{Hecke}_{\mathrm{inf,glob},x} \to \mathrm{Hecke}_{\mathrm{inf,glob},X}$$

resulting in forgetful functors

$$\mathrm{DMod}(\mathrm{Bun}_G) \to \mathcal{O}_{\mathrm{Hecke}_{\mathrm{inf,glob},x}}\text{-mod}(\mathrm{IndCoh}(\mathrm{Bun}_G)) \to \mathrm{IndCoh}(\mathrm{Bun}_G)$$

and their left adjoints (induction functors).

Now, observe that the formation of $\mathrm{Hecke}_{\mathrm{inf,glob},x}$ could be parametrized over x, so we get an enhanced version

$$\mathrm{Hecke}_{\mathrm{inf,glob,Ran_{dR}}} \rightrightarrows \mathrm{Bun}_G \times \mathrm{Ran_{dR}}.$$

This sits below the groupoid

$$\mathrm{Hecke}_{\mathrm{inf,glob},X} \times \mathrm{Ran_{dR}} \rightrightarrows \mathrm{Bun}_G \times \mathrm{Ran_{dR}}$$

from which we obtain an adjunction

$$\mathrm{ind}_{\mathrm{inf}\to D} : \mathcal{O}_{\mathrm{Hecke}_{\mathrm{inf,glob,Ran_{dR}}}}\text{-mod}(\mathrm{IndCoh}(\mathrm{Bun}_G)\otimes \mathrm{DMod}(\mathrm{Ran})) \rightleftarrows$$

$$\mathcal{O}_{\mathrm{Hecke}_{\mathrm{inf,glob},X}\times \mathrm{Ran_{dR}}}\text{-mod}(\mathrm{IndCoh}(\mathrm{Bun}_G)\otimes \mathrm{DMod}(\mathrm{Ran})) \simeq \mathrm{DMod}(\mathrm{Bun}_G)\otimes \mathrm{DMod}(\mathrm{Ran}) : \mathrm{oblv}_{D\to \mathrm{inf}}$$

and our localization functor will be defined as

$$\mathrm{Loc} := \mathrm{KL}(G)_{\mathrm{Ran}} \xrightarrow{\text{pullback}} \mathcal{O}_{\mathrm{Hecke}_{\mathrm{inf,glob,Ran_{dR}}}}\text{-mod}(\mathrm{IndCoh}(\mathrm{Bun}_G)\otimes \mathrm{DMod}(\mathrm{Ran}))$$

$$\xrightarrow{\mathrm{ind}_{\mathrm{inf}\to D}} \mathrm{DMod}(\mathrm{Bun}_G)\otimes \mathrm{DMod}(\mathrm{Ran}) \xrightarrow{\int_X} \mathrm{DMod}(\mathrm{Bun}_G).$$

Now, by considering the trivial groupoid we also have a functor

$$\mathrm{Loc}^+ := \mathrm{Rep}(\mathcal{L}^+G)_{\mathrm{Ran}} \to \mathrm{IndCoh}(\mathrm{Bun}_G).$$

The following is tautological:

$$\mathrm{ind}_{\mathrm{IndCoh}\to\mathrm{DMod}} \circ \mathrm{Loc}^+ \simeq \mathrm{Loc}\circ \mathrm{ind}_{\mathcal{L}^+G\to\mathrm{KL}}$$

The much more interesting fact is that

Proposition 4.1.3. *If A is a unital factorization algebra in $\mathrm{KL}(G)_{\mathrm{Ran}}$, then*

$$\mathrm{Loc}^+\circ \mathrm{oblv}_{\mathrm{KL}\to\mathcal{L}^+G}(A) \simeq \mathrm{oblv}_{\mathrm{DMod}\to\mathrm{IndCoh}} \circ \mathrm{Loc}(A).$$

The description of fiber in terms of A-conformal blocks, given above, is the output of this in the factorization module setting. We warn that this isomorphism *does not hold* if A is not a unital factorization algebra.

Intuition Why is the above statement true? Observe that $\mathrm{Hecke}_{\mathrm{inf,glob,Ran_{dR}}}$ has a factorization structure in the obvious sense, resulting in a *unital factorizable* monad

$$\mathcal{H} \in \mathrm{End}(\mathrm{IndCoh}(\mathrm{Bun}_G))\otimes \mathrm{DMod}(\mathrm{Ran});$$

roughly speaking, the endomorphism $\mathcal{F}_{x_1,\ldots,x_k}$ associated with k disjoint points is the composition $\mathcal{F}_{x_1}\circ\ldots\circ\mathcal{F}_{x_k}$.

Remark 4.1.4. Of course, this notion is problematic because $\mathrm{End}(\mathrm{IndCoh}(\mathrm{Bun}_G))$ is only monoidal yet $\{x_1,\ldots,x_k\}$ is unordered. For our Hecke modifications this is un-ambiguous, because Hecke modifications at distinct points commute with each other. In general, to define this notion one should use the ordered version of the Ran space.

The chiral homology of a unital factorizable monad $\mathcal{A}$ is an usual monad on $\mathrm{IndCoh}(\mathrm{Bun}_G)$ (this is not true without the phrase "unital factorizable"). Likewise, the chiral homology of a factorization-compatible[14] module over a factorization monad still carries the structure of a module over $\int_X \mathcal{A}$. The key fact proven by Rozenblyum is that

Proposition 4.1.5. $\int_X \mathcal{H} \simeq (\mathrm{oblv}_{\mathrm{DMod}\to\mathrm{IndCoh}}\circ \mathrm{ind}_{\mathrm{IndCoh}\to\mathrm{DMod}}) \in \mathrm{End}(\mathrm{IndCoh}(\mathrm{Bun}_G)).$

[14]We are being intentionally vague about what this amounts to.

A factorization algebra in $\mathrm{KL}(G)_{\mathrm{Ran}}$, upon pulling its underlying $\mathrm{Rep}(\mathcal{L}^+G)_{\mathrm{Ran}}$-object to $\mathrm{IndCoh}(\mathrm{Bun}_G)$, would be a factorization-compatible module over $\mathcal{H}$. It follows that chiral homology gives rise to a functor

$$\mathrm{FactAlg}_{\mathrm{un}}(\mathrm{KL}(G)_{\mathrm{Ran}}) \to \int_X \mathcal{H}\text{-mod}(\mathrm{IndCoh}(\mathrm{Bun}_G)) \simeq \mathrm{DMod}(\mathrm{Bun}_G)$$

which is compatible with the forgetful functor to $\mathrm{Rep}(\mathcal{L}^+G)$ and $\mathrm{IndCoh}(\mathrm{Bun}_G)$. We claim that this functor coincides with our Loc constructed above. In other words, the D-module structure "comes along for free" for factorization algebras.

Loose Ends There are a few things we've swept under the rug. For one, Bun_G is not quasi-compact so our renormalization procedure doesn't handle it on the nose; instead one should do the procedure above for each quasi-compact open substack and the take the limit.

More importantly, we have completely omitted the discussion of unitality. Most of it is bookkeeping following the motto that "ind-direction is unital, pro-direction is co-unital, anything that has both is corr-unital". Some details:

- Local infinitesimal Hecke stack $\mathrm{Hecke}_{\mathrm{inf}}$ and its higher self fiber products are cosimplicial systems of corr-unital spaces linked by strictly[15] unital morphisms;
- the same goes for global infinitesimal Hecke stacks, but note that the map $\mathrm{Hecke}_{\mathrm{inf,glob,Ran}} \to \mathrm{Hecke}_{\mathrm{inf}}$ is strictly co-unital;
- the localization functor Loc is a unital map between unital factorization categories, but does not respect the factorization structure; the map Loc^+ *does* respect this structure.

The Main Diagram Let us go back to our main diagram. We'll prove it in two parts. First let's prove that

$$\begin{tikzcd}
\mathrm{KL}_\kappa(G) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{Loc}"] & \mathrm{res}(\mathrm{Vac})\text{-}\mathrm{mod}^{\mathrm{Fact}}_{\mathrm{un}}(\mathrm{KL}_\kappa(B)) \arrow[d, "\mathrm{Loc}_{\mathrm{res}(\mathrm{Vac})}"] \\
\mathrm{DMod}_\kappa(\mathrm{Bun}_G) \arrow[r, "!\text{-pull}"] & \mathrm{DMod}_\kappa(\mathrm{Bun}_B)
\end{tikzcd}$$

The existence of a natural transformation from top circuit to the bottom follows from our geometric construction. To prove it's invertible, it suffices to append by the conservative map $\mathrm{oblv} : \mathrm{DMod}_\kappa(\mathrm{Bun}_B) \to \mathrm{IndCoh}(\mathrm{Bun}_B)$. By (variant) of Proposition 4.1.3, it suffices to prove the Loc^+ version of this diagram, which reads

$$\begin{tikzcd}
\mathrm{Rep}(\mathcal{L}^+G) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{Loc}^+"] & \mathrm{Rep}(\mathcal{L}^+B) \arrow[d, "\mathrm{Loc}^+"] \\
\mathrm{IndCoh}(\mathrm{Bun}_G) \arrow[r, "!\text{-pull}"] & \mathrm{IndCoh}(\mathrm{Bun}_B)
\end{tikzcd}$$

which is now just a base change.

[15]The word "strictly" always boils down to checking certain squares are Cartesian.

At this point we'll have to switch to the $*$-version in order to perform $*$-push. The following diagram is easy:

$$\begin{tikzcd} A\text{-mod}^{\mathrm{Fact}}_{\mathrm{un}}(\mathrm{KL}_\kappa(B)) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Loc}_A"] & A\text{-mod}^{\mathrm{Fact}}_{\mathrm{un}}(\mathrm{KL}^*_{\kappa'}(B)) \arrow[d, "\mathrm{Loc}_{\mathrm{co},A}"] \\ \mathrm{DMod}_\kappa(\mathrm{Bun}_B) \arrow[r, "\bullet\otimes K_{\mathrm{Bun}_B}"] & \mathrm{DMod}_{\kappa'}(\mathrm{Bun}_B) \end{tikzcd}$$

where

- A is any unital factorization algebra;
- κ' is the shift of κ by the critical level[16] and $\mathrm{Loc}_{\mathrm{co}}$ is an appropriately defined analogue of Loc;
- K_{Bun_B} is the dualizing sheaf of Bun_B.

Our last diagram is the following: for any unital factorization algebra A, the diagram

$$\begin{tikzcd} A\text{-mod}^{\mathrm{Fact}}_{\mathrm{un}}(\mathrm{KL}^*_{\kappa'}(B)) \arrow[r, "C^{\infty/2}_*(\mathcal{L}\mathfrak{n},-)"] \arrow[d, "\mathrm{Loc}_{\mathrm{co},A}"] & \Omega_A\text{-mod}^{\mathrm{Fact}}_{\mathrm{un}}(\mathrm{KL}^*_{\kappa'}(T)) \arrow[d, "\mathrm{Loc}_{\mathrm{co},\Omega_A}"] \\ \mathrm{DMod}_{\kappa'}(\mathrm{Bun}_B) \arrow[r, "*\text{-push}"] & \mathrm{DMod}_{\kappa'}(\mathrm{Bun}_T) \end{tikzcd}$$

commutes, where $\Omega_A := C^{\infty/2}_*(\mathcal{L}\mathfrak{n}, A)$. We can again make a quick series of reductions:

- Using the propagation method, we can replace A with the unit of $\mathrm{KL}^*_{\kappa'}(B)$ at the cost of working with $\mathrm{Ran}_{\mathrm{Ran}}$ (we'll show only the Ran version below);
- Because the induction functor $\mathrm{Rep}(\mathcal{L}^+B) \to \mathrm{KL}^*_{\kappa'}(B)$ generates the latter, it suffices to pre-compose with the induction functors. Since everything is $*$-pushforward, one immediately reduces to the following $+$-version of the diagram:

$$\begin{tikzcd} \mathrm{Rep}(\mathcal{L}^+B)_{\mathrm{Ran}} \arrow[r, "C^*(\mathcal{L}^+\mathfrak{n},-)"] \arrow[d, "\mathrm{Loc}^+"] & \mathrm{Rep}(\mathcal{L}^+T)_{\mathrm{Ran}} \arrow[d, "\mathrm{Loc}^+"] \\ \mathrm{IndCoh}(\mathrm{Bun}_B) \arrow[r, "*\text{-push}"] & \mathrm{IndCoh}(\mathrm{Bun}_T) \end{tikzcd}$$

- Because $\mathrm{Rep}(\mathcal{L}^+B)_{\mathrm{Ran}}$ is generated by images of the restriction map $\mathrm{Rep}(\mathcal{L}^+T)_{\mathrm{Ran}} \to \mathrm{Rep}(\mathcal{L}^+B)_{\mathrm{Ran}}$ it suffices to pre-compose with this restriction map. But everything in sight is $\mathrm{Rep}(\mathcal{L}^+T)_{\mathrm{Ran}}$-linear (here we use the fact that $\mathrm{Rep}(\mathcal{L}^+T)$ is a commutative factorization category), so it suffices to verify the diagram on the monoidal unit.

We will demonstrate this by verifying it at the trivial T-bundle on Bun_T; the actual proof is an easy generalization. Over the trivial T-bundle, our job is to verify the following diagram (where a top-to-bottom natural transformation is already given) over the monoidal unit:

$$\begin{tikzcd} \mathrm{Rep}(\mathcal{L}^+N)_{\mathrm{Ran}} \arrow[r, "C^*(\mathcal{L}^+\mathfrak{n},-)"] \arrow[d, "\mathrm{Loc}^+"] & \mathrm{DMod}(\mathrm{Ran}) \arrow[d, "\int_X"] \\ \mathrm{IndCoh}(\mathrm{Bun}_N) \arrow[r, "*\text{-push}"] & \mathrm{Vect} \end{tikzcd}$$

[16] This is at a point; as we mentioned in last week's talk, over the entire curve there is an additional anomaly twist.

In other words, we are to verify that the chiral homology of the commutative factorization algebra $C^*(\mathcal{L}^+\mathfrak{n}, k)$ is the cohomology of Bun_N.

Now, commutative chiral algebras are equivalent to commutative algebra objects in $\mathrm{DMod}(X)$ via restriction to the main diagonal; in such case, the chiral homology becomes the functor C_∇ of *algebra of horizontal sections*, which is the left adjoint to the functor

$$\bullet \otimes \omega_X : \mathrm{CAlg}(\mathrm{Vect}) \to \mathrm{CAlg}(\mathrm{DMod}(X));$$

(we remind this left adjoint is not computed via de Rham global section). So our goal is to show that

$$C_\nabla(C^*(\mathcal{L}^+\mathfrak{n}, k)_X) \simeq C^*(\mathrm{Bun}_N).$$

Now, LHS is computable: this is the fact (observed by Beilinson and Drinfeld) that the chiral homology of the Koszul dual of a Lie-$\star$ algebra is simply the Chevalley cochain of its global section. More explicitly, we have

$$C_\nabla(C^*(\mathcal{L}^+\mathfrak{n}, k)_X) \simeq C_\nabla(\mathrm{KD}(\mathfrak{n} \otimes D_X)_X) \simeq C^*(\Gamma(X_{\mathrm{dR}}, \mathfrak{n} \otimes D_X)).$$

This in particular shows that both $C_\nabla(C^*(\mathcal{L}^+\mathfrak{n}, k)_X)$ and $C_*(\mathrm{Bun}_N)$ are concentrated in cohomological degree ≥ 0. Now, it is a result of Lurie that the category of commutative algebras in degree ≥ 0 (known as *coaffine* algebras) embed into the category of prestacks by taking their spectrum, so it suffices to check that the corresponding prestacks match up. That of RHS is just Bun_N, whereas the LHS is, by definition of C_∇, the prestack classifying *horizontal sections* of $(\mathbb{B}\mathcal{L}^+N)_{X_{\mathrm{dR}}} \to X_{\mathrm{dR}}$, which is nothing but Bun_N.

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