1 Overview Talk by David Yang (Sep 17)

For this talk, $G = \mathrm{GL}_n$, B are the upper triangular matrices, N are the strictly upper triangular ones. Let's work over $\mathbb{C}$ for now. Consider $G(\mathbb{F}_q)$. How can we produce representations? One way is via induction:

$$V := \mathrm{Ind}_{B(\mathbb{F}_q)}^{G}\mathbb{C} \simeq \mathbb{C} \otimes_{\mathbb{C}[B]} \mathbb{C}[G]$$

then $\dim(V) = \dfrac{|G|}{|B|} \sim q!^{\frac{n^2-n}{2}}$. Let $V = \bigoplus_{\text{irrep}} V_i^{\oplus n_i}$, then $\mathrm{End}(V) \simeq \bigoplus \mathrm{End}(V_i)$. That is, the V_i are classified by representations of $\mathrm{End}(V)$.

Let $H := \mathrm{End}(V)$. Then irrep W of H yields irrep $V \otimes_H W$ of $G(F_q)$. This generates some positive proportions of the irreps. In fact, H lies in a family H_q (q for any complex number, becomes the previous case when q is the size of $\mathbb{F}_q$), where H_1 is the group algebra of $S_n =: W$. H_q has the same representation theory as H_1 for q not a root of unity. Also, (more to come later), $\mathrm{End}(V)$ is the convolution algebra of functions on $B\backslash G/B$.

Now let's try to do the same for infinite-dimensional cases, i.e. p-adic groups, case of $G(\mathbb{Q}_p)$.

Complex representations of $G(\mathbb{Q}_p)$: why do we care? If you are a number theorist then there's local Langlands, which compares such with n-dimensional Galois reps (of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.

Let's try to induce. Consider $G(\mathbb{Z}_p)$. So fix $V = \mathrm{Ind}_{G(\mathbb{Z}_p)}^{G(\mathbb{Q}_p)}\mathbb{C}$. Then $\mathrm{End}(V) = \mathbb{C}[G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)]$. Let $\mathbb{C}[G(\mathbb{Q}_p)]$ denote the algebra of compactly supported locally constant functions on $G(\mathbb{Q}_p)$. (Note the lack of unit.) Reps of $G(\mathbb{Q}_p)$ are reps of this algebra. Multiplication is given by convolution:

$$f * g(x) = \int f(x/y)g(y)dy$$

There's a specific element e inside the group algebra which is $\mathbf{1}$ on $G(\mathbb{Z}_p)$ and 0 everywhere else. (From now on $K = \mathbb{Q}_p, O = \mathbb{Z}_p$.)

Exercise 1. *Show this is compactly supported and locally constant.*

Proposition 1. *$e^2 = e$ under correct normalization.*

Let $V = \mathbb{C}[G(\mathbb{Q}_p)]e$. Then $\mathrm{End}(V) = e\mathbb{C}[G(K)]e$, the ring of bi-$G(O)$-invariant functions on $G(K)$.

Proposition 2. *This is the same as $K^0(\mathrm{Rep}(\mathrm{GL}_n)) \simeq \mathbb{C}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]^{S_n}$. (Note this is commutative! Compare with the finite-dimensional case.)*

Let's denote this by H. Then reps of $G(\mathbb{Q}_p)$ appearing in this particular representation are in bijections of reps of H i.e. maximal ideals in H. Note that also $H = \mathbb{C}(\mathrm{GL}_n//_{\text{adjoint}}\mathrm{GL}_n)$. (This is the Hecke compatibility.)

What about Iwahori I, i.e. the preimage of B via $G(O) \to G(\mathbb{F}_p)$? Again we take $V = \mathrm{Ind}_I^{G(K)}\mathbb{C}$, let $H = \mathrm{End}(V)$. Want to understand representation theory of H.

Exercise 2. *For GL_2, show that $G(O)\backslash G(K)/G(O)$ are exactly these ones: $G(O)\begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix}G(O)$, $a \geq b$. This is analogous to the KAK decomposition for real groups.*

Exercise 3. *On the other hand, show that $I\backslash G(K)/I$ are exactly these ones: $I\begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix}I$ and $I\begin{pmatrix} 0 & p^a \\ p^b & 0 \end{pmatrix}I$.*

Reductive groups are, we remind, classified by root systems. There are extensions of them, those with infinitely many roots. (Examples drawn on board.) To define such a thing you want to fix a symmetric bilinear form. Note you don't actually need the entire thing to be non-degenerate. For $\mathrm{SL}_2(K)$, the bilinear form is

$$\langle (a, b), (c, d)\rangle := ac$$

One can check that each root still defines a reflection. This is the root system $\hat{A}_1$.

What is the subgroup I in this context? It's everything above the line of slope $-1/2+\epsilon$. (Picture drawn on board.) For SL_2, it's $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $a, b, d \in G(O)$, $c \in pG(O)$.

We have the Iwahori decomposition:

$$G(K) \simeq \bigsqcup_{w \in W^{\mathrm{aff}}} IwI$$

where W^{aff} is the affine Weyl group. Alternatively, it's given by the semi-direct product of the cocharacter lattice by the finie Weyl group. H again lies in a family H_q, q any complex number, where H_1 is $\mathbb{C}[W^{\mathrm{aff}}]$.

Let $\check{G}$ be the Langlands dual, which is also GL_n. Let $\check{\mathcal{B}} = \check{G}/\check{B}$ be the flag variety.

Proposition 3. *We have* $\mathbb{C}[G(O)\backslash G(K)/G(O)] \simeq K^0(\mathrm{Rep}(\check{G})) \simeq K_0^{\check{G}}(\mathrm{pt})$.

Let $\check{\mathcal{N}}$ be the variety of nilpotent elements in $\check{\mathfrak{g}}$.

Proposition 4 (Springer Resolution)**.** *Consider $T^*\check{\mathcal{B}}$, which is the collection of a borel along with an element in its nilradical. The map $\pi : T^*\check{\mathcal{B}} \to \check{\mathcal{N}}$ is a birational equivaence, $\check{G}$-equivariant, and semi-small, and a crepant resolution of singularities.*

Define $\check{\mathrm{St}} = T^*\check{\mathcal{B}} \times_{\check{\mathcal{N}}} T^*\check{\mathcal{B}}$.

Theorem 1.1 (Kazhdan-Lusztig)**.** *Let $\tilde{H} = K_0^{\check{G}\times\mathbb{C}^\times}(\check{\mathrm{St}})$, then it's acted on by $K_0^{\mathbb{C}^\times}(\mathrm{pt}) \simeq \mathbb{C}[x]$. (This action is on fibers.) Then $H = \mathbb{C}[I\backslash G(K)/I] \simeq \tilde{H} \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/(x-p)$.*

Sanity check on size. Fact: $\check{\mathrm{St}}$ has $|S_n| = n!$ irreducible components. One of them is the cotangent itself. $K_0^{\check{G}}(T^*\check{\mathcal{B}}) = K_0\check{G}(\check{\mathcal{B}}) = K_0^{\check{B}}(\mathrm{pt}) = K_0^{\check{T}}(\mathrm{pt}) = K_0(\mathrm{Rep}(\check{T})) = \mathbb{C}[\mathbb{Z}^{\mathrm{rank}(T)}]$.

This is a hard theorem but will fall out of the study during this seminar.

Application Here's one application: classification of irreps of H (Deligne-Langlands correspondence). For any element $\mathfrak{n} \in \check{\mathcal{N}}$, let $\mathcal{B}_{\mathfrak{n}} := T^*\check{B} \times_{\check{\mathcal{N}}} \{\mathfrak{n}\}$. Then you have commuting actions of $K_0^{\check{G}\times\mathbb{C}^\times}(\mathrm{St})$ and $K_0^{\check{G}\times\mathbb{C}^\times}(\mathrm{pt})$ on $K_0^{\check{G}\times\mathbb{C}^\times}(\mathcal{B}_{\mathfrak{n}})$. Then any character on RHS immediately gives reps of LHS by tensoring. Claim: this gives all irreps.

Now we categorify. For $G(O)$ case, natural outcome is the following:

$$\mathrm{Shv}(G(O)\backslash G(K)/G(O)) \simeq \mathrm{Rep}(\check{G})$$

This is geometric Satake (in the abelian case). What about the I case?

$$\mathrm{Shv}(I\backslash G(K)/I) \simeq \mathrm{IndCoh}^{\check{G}}(\check{\mathrm{St}})$$

(This only holds derivedly, in contrary to the above.)

Application to modular rep theory Take GL_n as an algebraic group over $\mathbb{F}_q$. Can consider some special reps, e.g. reps reduced from $\mathbb{Z}$. These are not irreducible (in general), so what are the irreps and their characters? Equivalently, what are the transition matrix?

First hint: what are the blocks? Answer: blocks are affine Weyl orbits. Example: SL_3 (character lattice drawn). Then the irreps that can appear are those with highest weight in the W^{aff}-orbit. Second hint: the multiplicies $[V_x : W_{x'}]$ are values of (periodic) KL-polynomials (only true if x, x' are small enough compared to p^2).

How to prove? Originally: Anderson-Jantzen-Soergel via the Lusztig triangle: (modular) to (quantum groups at roots of unity) to (affine). New proof: links all three to $D^b\mathrm{Coh}^{\check{G}}(T^*\mathcal{B})$. To affine: Roman's thing. To modular: reps of $G/\overline{\mathbb{F}_q}$ corresponds to that of $\mathfrak{g}/\overline{\mathbb{F}_q}$. Now use BB localization (BMR version: for large p, using derived categories, using crystalline differential operators). Precise statement: exists an Azumaya algebra $\mathcal{A}$ on T^*X such that $D_X^{\mathrm{crys}} = \pi_*\mathcal{A}$. So imparticular, $D^{\mathrm{crys}}(\mathcal{B}) \simeq \mathcal{A}\text{-mod}(T^*X)$. Theorem (BMR): $\mathbb{A}$ splits on formal neighborhood of Springer fibers. Remark: choosing a Springer fiber is the same as choosing a Frobenius character of $\mathfrak{g}$. (To quantum: ABG.)

To finish the job we need to identify the t-structures. This is where the exotic coherent / perverse t-structures come in.