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Centralizers and $\mathbb{E}_n$-modules

$f: A \to B$ morphism of $\mathbb{E}_n$-algebras

$\rightsquigarrow$ $\mathbb{E}_n$-algebra $Z_{\mathbb{E}_n}(f)$ universal w/r/t

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "1 \otimes \mathrm{id}_A"'] & & B \\ & Z_{\mathbb{E}_n}(f) \otimes A \arrow[ur, "\mathbb{E}_n\text{-algebra morphism}"'] & \end{tikzcd}$$

$Z_{\mathbb{E}_n}(A) := Z_{\mathbb{E}_n}(\mathrm{id}_A)$ has the structure of an $\mathbb{E}_{n+1}$-algebra.

E.g. $n=1$: $\mathbb{E}_1$-alg $=$ AssocAlg, and

$$Z_{\mathbb{E}_1}(f) = \mathrm{Hom}_{A \otimes A^{op}}(A, B).$$

This formula generalizes. Put

$$A\text{-mod}_{\mathbb{E}_1} := (A \otimes A^{op})\text{-mod}_L,$$

i.e. bimodules. Recall that $A\text{-mod}_L$ is an $\mathbb{E}_{n-1}$-category, and put

$$A\text{-mod}_{\mathbb{E}_n} := Z_{\mathbb{E}_{n-1}}(A\text{-mod}_L),$$

the category of $\mathbb{E}_n$-modules for A.

<u>Proposition</u> There is a canonical $\mathbb{E}_n$-equivalence

$$Z_{\mathbb{E}_n}(A)\text{-mod}_L \xrightarrow{\sim} Z_{\mathbb{E}_{n-1}}(A\text{-mod}_L) = A\text{-mod}_{\mathbb{E}_n}.$$

Returning to $f: A \to B$, we have:

<u>Proposition</u> There is a canonical isomorphism

$$Z_{\mathbb{E}_n}(f) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-mod}_{\mathbb{E}_n}}(A, B)$$

in $A\text{-mod}_{\mathbb{E}_n}$.

This might be hard to compute in general, but if $f: A \to B$ is a morphism of commutative algebras, then

$$\mathrm{Hom}_{A\text{-mod}_{\mathbb{E}_n}}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-mod}_L}(S^n \otimes A, B).$$

Here $S^n \otimes A$ denotes the colimit in $\mathbb{E}_\infty$-alg of the constant diagram $S^n \to \mathbb{E}_\infty$-alg with value A. It receives a map $A \to S^n \otimes A$ in $\mathbb{E}_\infty$-alg, assuming we have chosen a basepoint $pt \to S^n$, hence has a left A-module structure.

Casselman-Shalika formula

We recall the naïve geometric Satake functor.

$$\mathrm{Sat}_G^{\text{naïve}}: \mathrm{QCoh}(LS_{\check{G}}(D)) \to D(G(\mathcal{O}) \backslash G(K) / G(\mathcal{O}))$$
$$\| \; \mathrm{Sph}_G.$$

It is a t-exact monoidal factorization functor, or over a point an $\mathbb{E}_3$-monoidal functor. The LHS is actually $\mathbb{E}_\infty$: it is the factorization category attached to the symm. mon. cat. $\mathrm{Rep}(\check{G})$.

The functor Sat_G^{naive} is not an equivalence, even though it induces an equivalence on the hearts. However, we have:

<u>Theorem</u> (Frenkel, Gaitsgory, Vilonen) The composition of factorization functors

$$\mathrm{QCoh}(LS_{\check{G}}(D)) \xrightarrow{\mathrm{Sat}_G^{naive}} \mathrm{Sph}_G \xrightarrow[\text{vacuum}]{\text{act on}} \mathrm{Wh}(\mathrm{Gr}_G)$$

is an equivalence.

Let's explain the functor on the RHS. We have fixed nondegenerate $\nu: N^- \to \mathbb{G}_a$, which yields

$$\psi: N^{-,\omega}(K) \longrightarrow \mathbb{G}_a^{\omega}(K) \xrightarrow{res} \mathbb{G}_a$$

"Ω_K" as add. ind-grp scheme

We put $\chi := \psi^! \exp$ and

$$\mathrm{Wh}(\mathrm{Gr}_G) := D(\mathrm{Gr}_G)^{(N^{-,\omega}(K), \chi)}.$$

This requires a little work to define carefully, because $N^{-,\omega}(K)$ is highly infinite dim. Use the fact that it is the union of its group subschemes.

We drop the ω's for simplicity of notation. Recall that the $N(K)$-orbits $S^\lambda \subset \mathrm{Gr}_G$ are parameterized by $\lambda \in \Lambda$.

<u>Lemma</u> An object of $\mathrm{Wh}(\mathrm{Gr}_G)$ is supported on the $S^{-,\lambda}$ for $\lambda \in \Lambda^+$.

The lemma follows from the following

observation: that χ is nontrivial when restricted to the stabilizer in $N^-(K)$ of a point in $S^{-\lambda}$, unless $\lambda \in \Lambda^+$.

Note that there is a map

$$\mathrm{can}: S^{-0} \xrightarrow{\sim} N^-(K)/N^-(\mathcal{O}) \xrightarrow{\psi} \mathbb{G}_a,$$

and put $\mathcal{W}^0 = \mathrm{can}^! \exp$. This is the vacuum object of $Wh(Gr_G)$.

The theorem can be checked over a point because both $QCoh(LS_{\check{G}}(D))$ and $Wh(Gr_G)$ are generated by ULA objects.

It will follow from:

Proposition We have
i) for any $\lambda \in \Lambda^+$ the object
$$\mathcal{W}^\lambda := Sat_G^{naive}(V^\lambda) * \mathcal{W}^0$$
has 1-dimensional !-fibers over $S^{-\lambda}$, and zero !- and $*$-fibers off $S^{-\lambda}$
ii) the category $Wh(Gr_G)$ is semisimple (in particular, equivalent to the derived category of its heart), and the irreducible objects of $Wh(Gr_G)^\heartsuit$ are precisely the $\mathcal{W}^\lambda$ for $\lambda \in \Lambda^+$.

Proof (i): Recall that $Sat_G^{naive}(V^\lambda) = IC_{\overline{Gr}_G^\lambda}$. This implies that $\mathcal{W}^\lambda$ is supported on the closure of $S^{-\lambda}$. See FGV...

mostly (i) $\Rightarrow$ (ii).

Derived Satake equivalence

$\mathcal{C}$ = factorization category

$\rightsquigarrow \mathcal{C}\text{-mod}^{\text{fact}}$ 2-cat. of fact. module cat.'s

Claim There is an equivalence of 2-cat's.

$$\mathcal{C}\text{-mod}^{\text{fact}}_x \xrightarrow{\sim} \mathcal{C}_x\text{-mod}_{\mathbb{E}_2}$$

This implies that

$$\text{End}_{\mathcal{C}\text{-mod}^{\text{fact}}_x}(\mathcal{C}_x) \xrightarrow{\sim} Z_{\mathbb{E}_2}(\mathcal{C}_x)$$

Now let $\mathcal{Y}_{X_{dR}} \to X_{dR}$ be a crystal of (formally) smooth Artin stacks.

$\rightsquigarrow$ factorization prestacks

$$\mathcal{Y}(D) \hookrightarrow \mathcal{Y}(\overset{\circ}{D})$$

$\mathcal{Y}(D)$ = horizontal jets, $\mathcal{Y}(\overset{\circ}{D})$ = meromorphic horizontal jets

Quasi-theorem (Lurie) There is an equivalence of monoidal factorization categories

$$\underset{\text{monoidal under convolution}}{\text{QCoh}(\mathcal{Y}(D) \times_{\mathcal{Y}(\overset{\circ}{D})} \mathcal{Y}(D))} \xrightarrow{\sim} \text{End}_{\text{QCoh}(\mathcal{Y}(D))\text{-mod}^{\text{fact}}}(\text{QCoh}(\mathcal{Y}(D)))$$

In the case $\mathcal{Y} = pt/G \times X_{dR}$, we write

$$LS_G(D) := \mathcal{Y}(D), \quad LS_{\check{G}}(\mathring{D}) := \mathcal{Y}(\mathring{D}).$$

Here the quasi-theorem is a theorem of Sam Raskin.

Now let's apply Casselman-Shalika. The action $Sph_G \curvearrowright Wh(Gr_G)$ is compatible with factorization. In particular it defines a functor

$$\begin{aligned} Sph_G &\longrightarrow End_{Wh(Gr_G)\text{-}mod^{fact}}(Wh(Gr_G)) \\ &\xrightarrow{\sim} End_{QCoh(LS_{\check{G}}(\mathring{D}))\text{-}mod^{fact}}(LS_{\check{G}}(D)) \\ &\xrightarrow[\sim]{Sam} QCoh(LS_{\check{G}}(D) \times_{LS_{\check{G}}(\mathring{D})} LS_{\check{G}}(D)). \quad (*) \end{aligned}$$

This functor is not yet an equiv. We must renormalize. Let $Sph_G^{loc.c.}$ be the full subcategory of objects whose image under $oblv_{G(O)}: Sph_G \to D(Gr_G)$ is compact. Let

$$Sph_G^{ren} := Ind(Sph_G^{loc.c.}) \rightleftarrows Sph_G$$

<u>Theorem</u> (Derived Satake) The functor $(*)$ fits into

$$\begin{tikzcd} Sph_G^{ren} \arrow[r, "Sat_G", "\sim"'] \arrow[d] & Ind Coh(LS_{\check{G}}(D) \times_{LS_{\check{G}}(\mathring{D})} LS_{\check{G}}(D)) \arrow[d, "\Phi"] \\ Sph_G \arrow[r, "(*)"] & QCoh(LS_{\check{G}}(D) \times_{LS_{\check{G}}(\mathring{D})} LS_{\check{G}}(D)) \end{tikzcd}$$

Φ – exists because spectral Hecke stack is ev. coconn.

Recall that Φ is the right adjoint

of the fully faithful functor Υ. There are many full subcat.'s between $\mathcal{Q}\mathrm{Coh}$ and IndCoh defined by singular support conditions.

In fact, the image of

$$\mathrm{Sph}_G \hookrightarrow \mathrm{Sph}_G^{ren} \xrightarrow{\sim} \mathrm{IndCoh}(LS_{\check{G}}(D) \times_{LS_{\check{G}}(D^\times)} LS_{\check{G}}(D))$$

is defined by the condition of sing. support in the nilpotent cone

$$\check{\mathcal{N}}/\check{G} \hookrightarrow \check{\mathfrak{g}}/\check{G},$$

see Arinkin-Gaitsgory for details.

One can reduce the theorem to calculation over a point using ULA generation properties.

<u>Proposition</u> Suppose $F: \mathcal{C} \to \mathcal{D}$ is a continuous functor between DG categories, and that $\{c_i\}$ is a set of compact generators for $\mathcal{C}$. If

i) $\{F(c_i)\}_i$ is a set of compact generators for $\mathcal{D}$, and
ii) we have
$$\mathrm{Hom}_{\mathcal{C}}(c_i, c_j) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F(c_i), F(c_j))$$
for all i, j.

then F is an equivalence of categories.

We will apply this with $\mathcal{C} = \mathrm{Sph}_G^{ren}$, which has the set of compact generators

$$\{\mathrm{Sat}_G^{\text{naive}}(V^\lambda) = \mathrm{Sat}_G^{\text{naive}}(V^\lambda) \star \delta_1\}_{\lambda\in\Lambda^+}$$

Here δ_1 is the direct image under

$$pt/G(\mathcal{O}) \xhookrightarrow{1} G(\mathcal{O})\backslash G(K)/G(\mathcal{O})$$

of $pt/G(\mathcal{O})$. Note that δ_1 belongs to $\mathrm{Sph}_G^{\text{loc.c}}$, but is not compact in Sph_G. By Casselman-Shalika and the construction of $(*)$,

$$\begin{tikzcd}
 & \mathrm{Sph}_G \arrow[d, "(*)"] \\
Q\mathrm{Coh}(\mathrm{LS}_{\check G}(D)) \arrow[ur, "\mathrm{Sat}_G^{\text{naive}}"] \arrow[r, "\Delta_*"'] & \mathcal{O}(\mathrm{Coh}(\mathrm{LS}_{\check G}(D) \times_{\mathrm{LS}_{\check G}(\overset{\circ}{D})} \mathrm{LS}_{\check G}(D)))
\end{tikzcd}$$

This implies that $(*)$ applied to $\mathrm{Sat}_G^{\text{naive}}(V^\lambda)$ is $\Delta_*(V^\lambda)$, and these indeed form a set of compact generators for $\mathrm{IndCoh}(\mathrm{LS}_{\check G}(D) \times_{\mathrm{LS}_{\check G}(\overset{\circ}{D})} \mathrm{LS}_{\check G}(D))$ after applying I.

In particular we can define Sat_G as the ind-extension of

$$\mathrm{Sph}_G^{\text{loc.c.}} \xrightarrow{(*)} \mathrm{Coh}(\mathrm{LS}_{\check G}(D) \times_{\mathrm{LS}_{\check G}(\overset{\circ}{D})} \mathrm{LS}_{\check G}(D)).$$

It remains to check (ii) in the prop. For this we use:

<u>Proposition</u> There is a canonical monoidal equivalence

$$\begin{tikzcd}
\mathrm{IndCoh}(\mathrm{LS}_{\check G}(D) \times_{\mathrm{LS}_{\check G}(\overset{\circ}{D})} \mathrm{LS}_{\check G}(D)) \arrow[r, "KD", "\sim"'] \arrow[d, shift right, "\Delta_*"'] & \mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}^{\check G} \arrow[d, shift left, "\mathrm{oblv}"] \\
\mathrm{IndCoh}(\mathrm{LS}_{\check G}(D)) \arrow[u, shift right, "\Delta^!"'] \arrow[r, "\sim"] & \mathrm{Rep}(\check G) \arrow[u, shift left, "\mathrm{ind}"]
\end{tikzcd}$$

A direct calculation shows that

$$
\begin{aligned}
\mathrm{End}_{\mathrm{Sph}_G}(\delta_1) &\xrightarrow{\sim} H^\bullet_{dR}(pt/G(\mathcal{O})) \\
&\xrightarrow{\sim} H^\bullet_{dR}(pt/G) \\
\text{Chern-Weil} \quad &\xrightarrow{\sim} \mathrm{Sym}(\mathfrak{t}^*[-2])^W \\
&\xrightarrow{\sim} \mathrm{Sym}(\check{\mathfrak{t}}[-2])^W \\
&\xrightarrow{\sim} \mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}}.
\end{aligned}
$$

Similarly, one checks that there is a canonical $\mathrm{Sym}(\check{\mathfrak{g}}[-2])^{\check{G}}$-linear isomorphism

$$
\mathrm{Hom}_{\mathrm{Sph}_G}(\delta_1, \mathrm{Sat}_G^{\text{naive}}(V)) \xrightarrow{\sim} (\mathrm{Sym}(\check{\mathfrak{g}}[-2]) \otimes V)^{\check{G}}.
$$

Finally, this implies that

$$
\begin{aligned}
\mathrm{Hom}_{\mathrm{Sph}_G}(\mathrm{Sat}_G^{\text{naive}}(V^\lambda), \mathrm{Sat}_G^{\text{naive}}(V^\mu)) &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sph}_G}(\delta_1, \mathrm{Sat}_G(V^{\lambda,\vee} \otimes V^\mu)) \\
&\xrightarrow{\sim} (\mathrm{Sym}(\check{\mathfrak{g}}[-2]) \otimes V^{\lambda,\vee} \otimes V^\mu)^{\check{G}} \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{-mod}^{\check{G}}}(\mathrm{Sym}(\check{\mathfrak{g}}[-2]) \otimes V^\lambda, \mathrm{Sym}(\check{\mathfrak{g}}[-2]) \otimes V^\mu) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{IndCoh}(LS_{\check{G}}(\mathcal{D}) \times_{LS_{\check{G}}(\mathring{\mathcal{D}})} LS_{\check{G}}(\mathcal{D}))}(\Delta_*(V^\lambda), \Delta_*(V^\mu)),
\end{aligned}
$$

as desired.