Factorization categories

crystal of algebraic (ind-)stacks

$$\begin{tikzcd} \mathcal{Y}_{X_{dR}} \arrow[r, hook] \arrow[d] & \mathcal{Y}_{\mathrm{Ran}_{dR}} \arrow[d] \\ X_{dR} \arrow[r, hook] & \mathrm{Ran}_{dR} \end{tikzcd}$$

$\mathcal{Y}_{\mathrm{Ran}_{dR}} = \mathrm{Jets}_{\mathrm{horiz}}(\mathcal{Y}_{X_{dR}})$ factorization space

$\rightsquigarrow D(\mathrm{Ran}) \circlearrowright \mathrm{IndCoh}(\mathcal{Y}_{\mathrm{Ran}_{dR}})$

More precisely, there is a sheaf of categories $\mathrm{IndCoh}(\mathcal{Y})$ on Ran_{dR}. Passing to global sections yields the action above.

$$\begin{tikzcd} & (\mathrm{Ran}_{dR} \times \mathrm{Ran}_{dR})_{disj} \arrow[dl, "j"'] \arrow[dr, "add \circ j"] & \\ \mathrm{Ran}_{dR} \times \mathrm{Ran}_{dR} \arrow[rr, "add"] & & \mathrm{Ran}_{dR} \end{tikzcd}$$

__Def'n__ A __factorization category__ on X is a sheaf of categories $\mathcal{F}$ on Ran_{dR} equipped with equivalences

$$(add \circ j)^* \mathcal{F} \xrightarrow{\sim} j^*(\mathcal{F} \boxtimes \mathcal{F}).$$

and homotopy-coherent compatibilities.

For $\mathcal{Y}$ as above, $\mathrm{IndCoh}(\mathcal{Y})$ forms a factorization category on X.

Q: fix $x \in X(k)$. What structure does $\mathrm{IndCoh}(\mathcal{Y}_x)$ inherit from factorization?

$\mathbb{E}_n$-algebras

$\mathcal{A}$ = symmetric monoidal category

$\rightsquigarrow$ symmetric monoidal structure on $\mathbb{E}_1\text{-alg}(\mathcal{A})$

<u>Def'n</u> The category of <u>$\mathbb{E}_n$-algebras in</u> is

$$\mathbb{E}_n\text{-alg}(\mathcal{A}) := \underbrace{\mathbb{E}_1\text{-alg}(\cdots \mathbb{E}_1}_{n \text{ times}}\text{-alg}(\mathcal{A}) \cdots)$$

In particular, we can take $\mathcal{A} = \mathrm{DGCat}$ to obtain the category of <u>$\mathbb{E}_n$-categories</u>.

<u>Exercise</u> Take $\mathcal{A}$ to be the category of $(1,1)$-categories with the Cartesian symm. monoidal structure. Then

$$\mathbb{E}_2\text{-alg}(\mathcal{A}) \xrightarrow{\sim} \{\text{braided monoidal } (1,1)\text{-cat's}\}.$$

and

$$\mathbb{E}_n\text{-alg}(\mathcal{A}) \xrightarrow{\sim} \{\text{symm. monoidal } (1,1)\text{-cat.'s}\}$$

for $n > 3$.

$\mathcal{C} = \mathbb{E}_n$-category

$\rightsquigarrow \mathbb{E}_{n-1}$-monoidal structure on $\mathbb{E}_1\text{-alg}(\mathcal{C})$

So we can speak of $\mathbb{E}_k$-algebras in $\mathcal{C}$ for all $k \leq n$.

$$\mathbb{E}_\infty\text{-alg}(\mathcal{A}) := \mathrm{ComAlg}(\mathcal{A}) \xrightarrow{\mathrm{res}^{\mathbb{E}_\infty}_{\mathbb{E}_n}} \mathbb{E}_n\text{-alg}(\mathcal{A})$$

Back to factorization

$\mathcal{C}$ = factorization category

For technical reasons we need some hypotheses on $\mathcal{C}$. We can either assume:

i) $\mathcal{C}$ is generated by ULA objects.
  or, more stringently,
ii) $X = \mathbb{A}^1$ and $\mathcal{C}$ is translation-invariant.

<u>Quasi-theorem</u> For any $x \in X(k)$, the category $\mathcal{C}_x := x^*\mathcal{C}$ has a natural E_2-monoidal structure under fusion.

Here's the idea. For simplicity we assume $X = \mathbb{A}^1$, and that $\mathcal{C} = \mathrm{IndCoh}(\mathcal{Y})$ where

$$\mathcal{Y}_{\mathbb{A}^1_{dR}} \cong \mathcal{Y}_0 \times \mathbb{A}^1_{dR}$$

noncanonically, so that we are in situation (ii).

For any $(t_1, t_2) \in \mathbb{A}^2 \setminus \Delta(\mathbb{A}^1)$ we get

$$\mathbb{A}^1 \longrightarrow \mathbb{A}^2$$
$$s \longmapsto (st_1, st_2)$$

Base-changing $\mathcal{Y}_{\mathbb{A}^2_{dR}}$ along this map yields a degeneration of $\mathcal{Y}_0 \times \mathcal{Y}_0$ to $\mathcal{Y}_0$. The <u>fusion</u> of $\mathcal{F}$ and $\mathcal{G}$ in $\mathrm{IndCoh}(\mathcal{Y}_0)$ is

$$\mathcal{F} \odot \mathcal{G} := 0^! \underbrace{\Psi}_{\text{nearby cycles}}\left((\mathcal{F} \boxtimes \mathcal{G} \boxtimes \omega_{\mathbb{A}^2_{dR}})|_{\mathcal{Y}_{\mathbb{A}^2_{dR}/\Delta}}\right).$$

Note that if $k = \mathbb{C}$, then the choice of $(t_1, t_2) \in \mathbb{A}^2 \setminus \Delta$ has the homotopy type of S^1. Thus we have an S^1-family of monoidal structures on $\mathrm{IndCoh}(\mathcal{Y}_0)$, as one expects from an $\mathbb{E}_2$-category.

We also have the following variant: suppose $\mathcal{C}$ is a monoidal factorization category. This means $\mathcal{C}$ is a sheaf of monoidal DG categories on Ran_{dR}, equipped with a factorization structure.

For instance, if $\mathcal{Y}_{X_{dR}}$ has a groupoid structure then $\mathrm{IndCoh}(\mathcal{Y})$ is a monoidal fact. cat.

In this situation $\mathcal{C}_x$ has a natural $\mathbb{E}_3$-monoidal structure.

$$\begin{tikzcd} \{\text{symm. mon. DG cat.'s}\} \arrow[r, dashed, "\substack{\text{concrete}\\ \text{construction}}"] \arrow[dr, "\mathrm{res}^{\mathbb{E}_\infty}_{\mathbb{E}_2}"'] & \{\text{factorization cat's}\} \arrow[d, "x^*"] \\ & \{\mathbb{E}_2\text{-categories}\} \end{tikzcd}$$

For example, applying this to the symmetric monoidal category $\mathrm{IndCoh}(\mathcal{Y}_0)$ yields $\mathrm{IndCoh}(\mathcal{Y})$ where $\mathcal{Y} = \mathrm{Jets}_{\mathrm{horiz}}(\mathcal{Y}_0 \times X_{dR})$.

We will consider the case $\mathcal{Y}_0 = pt/G$. Then

$$\mathcal{Y} = \mathrm{LS}_G(D),$$

the factorization space of G-local systems on X, or rather on parameterized disks in X.

<u>Naïve Satake</u>

We apply the formalism developed above to

$$\begin{array}{c} Gr_{G, X_{dR}} \\ \downarrow \\ X_{dR} \end{array}$$

So $D(Gr_G)$ has a factorization structure. We can also consider the Hecke stack

$$\mathcal{H}_{G, X_{dR}} := \{x : S \to X_{dR},\ \mathcal{P}_G, \mathcal{P}_G' = G\text{-bundles on } X \times S,\ \alpha : \mathcal{P}_G|_{(X\times S) - \Gamma_x} \xrightarrow{\sim} \mathcal{P}_G'|_{(X\times S)-\Gamma_x}\}.$$

It is a groupoid acting on $Gr_{G, X_{dR}}$ over X_{dR}.

$$Sph_G := D(\mathcal{H}_G) \text{ monoidal factorization category}$$

<u>Theorem</u> (Naïve Satake) There is a monoidal functor of factorization categories

$$Sat_G^{naive} : QCoh(LS_{\check{G}}(D)) \longrightarrow Sph_{G,crit} = D_{crit}(\mathcal{H}_G)$$

Pointwise, this is an $\mathbb{E}_3$-monoidal functor

$$Sat_{G,x}^{naive} : Rep(\check{G}) \longrightarrow Sph_{G,crit,x}.$$

A key property of this functor is that it is t-exact and induces an equivalence on the hearts of the t-structures.

Now we will sketch the proof of the theorem. It proceeds by constructing

$$\mathrm{Rep}(\check{G})^{\heartsuit} \xrightarrow{\sim} \mathrm{Sph}^{\heartsuit}_{G,\mathrm{crit},x},$$

an equivalence of symmetric monoidal (1,1)-cat.'s. Here the RHS is naturally $\mathbb{E}_3$-monoidal because fusion is t-exact, hence symm. monoidal because it's a (1,1)-categories.

Using the fact that $\mathrm{Rep}(\check{G})$ is the derived cat. of $\mathrm{Rep}(\check{G})^{\heartsuit}$, this would yield an $\mathbb{E}_3$-monoidal functor

$$\mathrm{Sat}^{\mathrm{naive}}_{G,x} : \mathrm{Rep}(\check{G}) \longrightarrow \mathrm{Sph}_{G,\mathrm{crit},x}.$$

This is tantamount to constructing $\mathrm{Sat}^{\mathrm{naive}}_{G}$ by the quasi-theorems above.

We must construct a symmetric monoidal "fiber functor"

$$F : \mathrm{Sph}^{\heartsuit}_{G,x} \longrightarrow \mathrm{Vect}^{\heartsuit},$$

which we will show is continuous, exact, and conservative. Then it follows that

Tannakian theorem

$$\begin{tikzcd} \mathrm{Sph}^{\heartsuit}_{G,x} \arrow[rr, "\sim", "\text{symmetric monoidal}"'] \arrow[dr, "F"'] & & \mathrm{Rep}(\mathrm{Aut}^{\otimes}(F))^{\heartsuit} \arrow[dl, "\mathrm{oblv}"] \\ & \mathrm{Vect}^{\heartsuit} & \end{tikzcd}$$

A priori $\mathrm{Aut}^{\otimes}(F)$ is an affine group scheme.

The remainder of the proof, which we skip, is showing that $\mathrm{Aut}^{\otimes}(F) \cong \check{G}$.

Construction of the fiber functor:

$$\mathrm{Sph}_G \longrightarrow D(\mathrm{Gr}_G) \xrightarrow{H^{\bullet}_{dR}(\mathrm{Gr}_G, -)} \mathrm{Vect}$$

First we will prove that this functor is t-exact, up to some shifts, so that it induces an exact functor

$$\mathrm{Sph}_G^{\heartsuit} \longrightarrow \mathrm{Vect}^{\heartsuit}.$$

Consider the correspondences

$$\begin{tikzcd}
 & \mathrm{Gr}_B^{\lambda} \arrow[dl, "q^{\lambda}"'] \arrow[dr, "p^{\lambda}"] & \\
\mathrm{Gr}_T^{\lambda} & & \mathrm{Gr}_G \\
 & \mathrm{Gr}_{B^-}^{\lambda} \arrow[ul, "q^{-,\lambda}"] \arrow[ur, "p^{-,\lambda}"'] &
\end{tikzcd}$$

$\mathrm{Gr}^{\lambda}_{B_{red}} \xrightarrow{\sim} S^{\lambda}$

semi-infinite orbit (for $N(K)$-action)

<u>Proposition</u> There is a canonical isomorphism

$$(q^{\lambda})_*(p^{\lambda})^! \xrightarrow{\sim} (q^{-,\lambda})_!(p^{-,\lambda})^* \qquad (*)$$

of functors

$$D(\mathrm{Gr}_G)^{T\text{-mon.}} \longrightarrow D(\mathrm{Gr}_T^{\lambda}) = \mathrm{Vect}.$$

This is a standard application of Braden's theorem on $\mathbb{G}_m$-actions.

One shows using dim. estimates for the $S^\lambda \cap Gr_G^\mu$ that

$$(\eta^\lambda)_*(p^\lambda)^! : Sph_G \longrightarrow D(Gr_G)^{T\text{-mon}} \longrightarrow Vect.$$

is left t-exact up to shift by $\langle \lambda, 2\check{\rho} \rangle$. Verdier dual considerations show that $(\eta^{-\lambda})_!(p^{-\lambda})^*$ is right t-exact on Sph_G up to shift by $\langle \lambda, 2\check{\rho} \rangle$, which by (*) would imply that both functors are t-exact up to shift.

<u>Proposition</u> for $\mathcal{F}$ in Sph_G There are canonical isomorphisms

$$H^\bullet_{dR}(Gr_G, \mathcal{F}) \xrightarrow{\sim} \bigoplus_{\lambda \in \Lambda} H^\bullet(S^\lambda, (j^\lambda)^! \mathcal{F})$$

$$\xrightarrow{\sim} \bigoplus_{\lambda \in \Lambda} H^\bullet_c(S^{-\lambda}, (j^{-\lambda})^* \mathcal{F})$$

<u>Proof</u> The isomorphism (*) implies that the Cousin filtrations on $H^\bullet(Gr_G, \mathcal{F})$ induced by the S^λ and the $S^{-\lambda}$ split each other. $\square$

In particular we obtain from this an exact, continuous fiber functor

$$F : Sph^\heartsuit_{G,x} \longrightarrow Rep(\check{T})^\heartsuit \xrightarrow{oblv} Vect^\heartsuit$$

For the symmetric monoidal structure on F:

$$\begin{tikzcd} \mathcal{Y}_{X_{dR}} \arrow[rr, "f\text{ - ind-proper}"] \arrow[dr] & & \mathcal{Z}_{X_{dR}} \arrow[dl] \\ & X_{dR} & \end{tikzcd}$$

__Proposition__ There is an induced functor of factorization cat.'s

$$f_*: \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Z}).$$

It follows that $H_{dR}(\mathrm{Gr}_G, -)$ defines an $\mathbb{E}_2$-monoidal functor

$$\mathrm{Sph}_{G,\mathrm{crit},x} \longrightarrow \mathrm{Rep}(\check{T}),$$

hence an $\mathbb{E}_2$-monoidal functor

$$\mathrm{Sph}^{\heartsuit}_{G,\mathrm{crit},x} \longrightarrow \mathrm{Rep}(\check{T})^{\heartsuit}$$

after appropriate shifts. Since

$$\mathrm{Fun}_{\mathbb{E}_\infty\text{-}\mathrm{Cat}}(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathbb{E}_2\text{-}\mathrm{Cat}}(\mathcal{C}, \mathcal{D})$$

for symmetric monoidal $(1,1)$-categories $\mathcal{C}$ and $\mathcal{D}$, we obtain the desired symm. mon. structure.

Finally, we show that F is conservative. Note that if $\mathcal{F}$ in $D(\mathrm{Gr}_G)$ is such that $(j^\lambda)^! \mathcal{F}$ for all $\lambda \in \Lambda$, then $\mathcal{F} = 0$.

__Lemma__ Let $F: \mathcal{C}_0 \to \mathcal{D}_0$ be an exact conservative functor between small abelian categories. Then $\mathrm{Ind}(F): \mathrm{Ind}(\mathcal{C}_0) \to \mathrm{Ind}(\mathcal{D}_0)$ is conservative.

NB: This lemma fails for DG categories.

As a corollary, it suffices to prove

that our fiber functor F preserves compact objects, and is conservative on compact objects.

<u>Proposition</u> The compact objects in $\mathrm{Sph}_{G,x}$ are precisely the holonomic complexes.

In particular, a compact object in $\mathrm{Sph}_{G,x}$ is supported on a finite-dimensional substack of $\mathcal{H}_{G,x}$ (this is easier than the Prop.).

Let $\mathcal{M}$ be a compact object of $\mathrm{Sph}^{\heartsuit}_{G,x}$. Then $F(\mathcal{M})$ is finite-dimensional, because $\mathcal{M}$ is supported on a finite-dim. subscheme of Gr_G.

If $\mathcal{M} \neq 0$, then there exists $\lambda \in \Lambda^+$ such that Gr^λ_G is open in the support of $\mathcal{M}$. Then $S^\lambda \cap \mathrm{Gr}_G^{\lambda} = S^\lambda \cap \overline{\mathrm{Gr}}^\lambda_G$ is open in Gr_G and closed in S^λ. Moreover $\mathcal{M}$ is constant on Gr^λ_G, hence on $S^\lambda \cap \mathrm{Gr}^\lambda_G$. It follows that

$$H^{\bullet}_{dR}(S^\lambda, (j^\lambda)^! \mathcal{M}) \neq 0,$$

and we have

$$H^{\langle \lambda, 2\rho \rangle}_{dR}(S^\lambda, (j^\lambda)^! \mathcal{M}) \hookrightarrow F(\mathcal{M}).$$