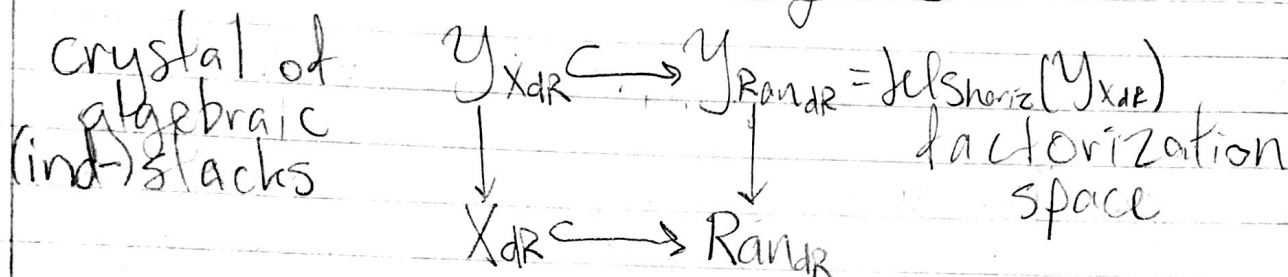


## Factorization categories



$$\rightsquigarrow D(\text{Ran}) \hookrightarrow \text{IndCoh}(\mathcal{Y}_{\text{Ran}_{dR}})$$

More precisely, there is a sheaf of categories  $\text{IndCoh}(\mathcal{Y})$  on  $\text{Ran}_{dR}$ . Passing to global sections yields the action above.

$$\begin{array}{ccc}
 & (\text{Ran}_{dR} \times \text{Ran}_{dR})_{\text{disj}} & \\
 j \swarrow & & \searrow \text{add} \circ j \\
 \text{Ran}_{dR} \times \text{Ran}_{dR} & \xrightarrow{\text{add}} & \text{Ran}_{dR}
 \end{array}$$

Def'n A factorization category on  $X$  is a sheaf of categories  $\mathcal{F}$  on  $\text{Ran}_{dR}$  equipped with equivalences

$$(\text{add} \circ j)^* \mathcal{F} \xrightarrow{\sim} j^*(\mathcal{F} \boxtimes \mathcal{F})$$

and homology-coherent compatibilities.

For  $\mathcal{Y}$  as above,  $\text{IndCoh}(\mathcal{Y})$  forms a factorization category on  $X$ .

Q: fix  $x \in X(k)$ . What structure does  $\text{IndCoh}(\mathcal{Y}_x)$  inherit from factorization?

## $E_n$ -algebras

$\mathcal{A}$ -symmetric monoidal category

$\rightsquigarrow$  symmetric monoidal structure on  $E_n\text{-alg}(\mathcal{A})$

Defn The category of  $E_n$ -algebras in      is

$$\rightarrow E_n\text{-alg}(\mathcal{A}) := \underbrace{E_1\text{-alg}(\dots E_1\text{-alg}(\mathcal{A}) \dots)}_{n \text{ times}}$$

In particular, we can take  $\mathcal{A} = \text{DGCat}$  to obtain the category of  $E_n$ -categories.

Exercise Take  $\mathcal{A}$  to be the category of  $(1,1)$ -categories with the cartesian symm. monoidal structure. Then

$$E_2\text{-alg}(\mathcal{A}) \xrightarrow{\sim} \text{braided monoidal } (1,1)\text{-cat. s.},$$

$$E_n\text{-alg}(\mathcal{A}) \xrightarrow{\sim} \text{symm. monoidal } (1,1)\text{-cat. s.}$$

for  $n > 3$ .

$\mathcal{L} = E_n$ -category

$\rightsquigarrow E_{n-1}$ -monoidal structure on  $E_1\text{-alg}(\mathcal{L})$

So we can speak of  $E_k$ -algebras in  $\mathcal{L}$  for all  $n \leq k$ .

$$E_0\text{-alg}(\mathcal{A}) := \text{ComAlg}(\mathcal{A}) \xrightarrow{\text{res}_{E_n}^{E_0}} E_n\text{-alg}(\mathcal{A})$$

## Back to factorization

$\mathcal{L}$  = factorization category

For technical reasons we need some hypotheses on  $\mathcal{L}$ . We can either assume:

- i)  $\mathcal{L}$  is generated by  $\mathcal{V}A$  objects.
- ii)  $X = A^{\otimes n}$  or more stringently,  $\mathcal{L}$  is fusion-invariant.

Quasi-theorem For any  $x \in X(k)$ , the category has a natural  $E_2$ -monoidal structure under fusion.

Here's the idea. For simplicity we assume  $X = A^{\otimes n}$  and that  $\mathcal{L} = \text{IndCoh}(\mathcal{Y})$  where

$$\mathcal{Y}_{\text{Aur}} \cong \mathcal{Y}_0 \times \mathbb{A}^1_{\text{Aur}}$$

noncanonically, so that we are in situation (i).

For any  $(t_1, t_2) \in A^2 \setminus \Delta(A^1)$  we get

$$\begin{array}{ccc} A^1 & \longrightarrow & A^2 \\ s_1 \mapsto & & (s_1, s_1 t_2) \end{array}$$

Base-changing  $\mathcal{Y}_{\text{Aur}}$  along this map  $\text{Fin}$  yields a degeneration of  $\mathcal{Y}_0 \times \mathcal{Y}_0$  to  $\text{IndCoh}(\mathcal{Y}_0)$  is fusion of  $\mathcal{Y}$  and  $\mathcal{Y}$  in nearby cycles.

$$\mathcal{Y} \otimes_{\mathcal{Y}} := \mathcal{D}^b \mathbb{P}((\mathcal{Y} \boxtimes \mathcal{Y}) \otimes_{\mathcal{Y}_{\text{Aur}}} \mathcal{Y}_{\text{Aur}}).$$



## Naïve Satake

We apply the formalism developed above to

$$\begin{array}{c} \text{Gr}_{G, X_{\text{dR}}} \\ \downarrow \\ X_{\text{dR}} \end{array}$$

so  $D(\text{Gr}_G)$  has a factorization structure. We can also consider the Hecke stack

$$\mathcal{M}_{G, X_{\text{dR}}} := \{x: D \rightarrow X_{\text{dR}}, P_G, P'_G = G\text{-bundles on } X \times S, \alpha: P_G(x \times S) \rightarrow P_G(x \times S) \times S\}$$

It is a groupoid acting on  $\text{Gr}_{G, X_{\text{dR}}}$  over  $X_{\text{dR}}$ .  
 $\text{Sph}_G := D(\text{He})$  monoidal factorization category

Theorem (Naïve Satake) There is a monoidal functor of factorization categories

$$\text{Sph}_G^{\text{naïve}}: \text{QCoh}(LSic(D)) \rightarrow \text{Sph}_G, \text{crit} = \text{Dcrit}(\text{He})$$

Poincaré, this is an  $E_3$ -monoidal functor

$$\text{Sph}_{\text{Gr}_x}^{\text{naïve}}: \text{Rep}(\hat{G}) \rightarrow \text{Sph}_G, \text{crit}_x.$$

A key property of this functor is that it is  $t$ -exact and induces an equivalence on the hearts of the  $t$ -structures.

Now we will sketch the proof of the theorem. It proceeds by constructing

$$\text{Rep}(\mathcal{G})^{\mathcal{R}} \xrightarrow{\sim} \text{Sph}_{\mathcal{G}, \text{ort}, x}^{\mathcal{R}}$$

an equivalence of symmetric monoidal  $(1,1)$ -cat's. Here the RHS is naturally  $\mathbb{F}_3$ -monoidal because fusion is  $f$ -exact, hence symmetric monoidal because it's a  $(1,1)$ -categories.

Using the fact that  $\text{Rep}(\mathcal{G})$  is the derived cat. of  $\text{Rep}(\mathcal{G})^{\mathcal{R}}$ , this would yield an  $\mathbb{F}_3$ -monoidal functor

$$\text{Saf}_{\mathcal{G}, x}^{\text{maine}}: \text{Rep}(\mathcal{G}) \longrightarrow \text{Sph}_{\mathcal{G}, \text{ort}, x}^{\mathcal{R}}$$

This is an attempt to construct  $\text{Saf}_{\mathcal{G}, x}^{\text{maine}}$  by the quasi-theorems above. We must construct all fiber functors "symmetric monoidal"

$$F: \text{Sph}_{\mathcal{G}, x}^{\mathcal{R}} \longrightarrow \text{Vect}_{\mathbb{F}_3}$$

which we will show is continuous, exact, and conservative. Then H follows that

$$\begin{array}{ccc} \text{Fannulation} & & \\ \text{theorem} & & \\ \text{Sph}_{\mathcal{G}, x}^{\mathcal{R}} & \xrightarrow{\text{symmetric monoidal}} & \text{Rep}(\text{Aut}_{\mathcal{R}}(\mathbb{F}_3))^{\mathcal{R}} \\ \downarrow F & & \downarrow \text{oblv} \\ \text{Vect}_{\mathbb{F}_3} & & \end{array}$$

$\text{Aut}_{\mathcal{R}}(\mathbb{F}_3)$  is an affine group scheme,



The remainder of the proof which we skip is showing that  $\text{Aut}(F) \cong G$ .

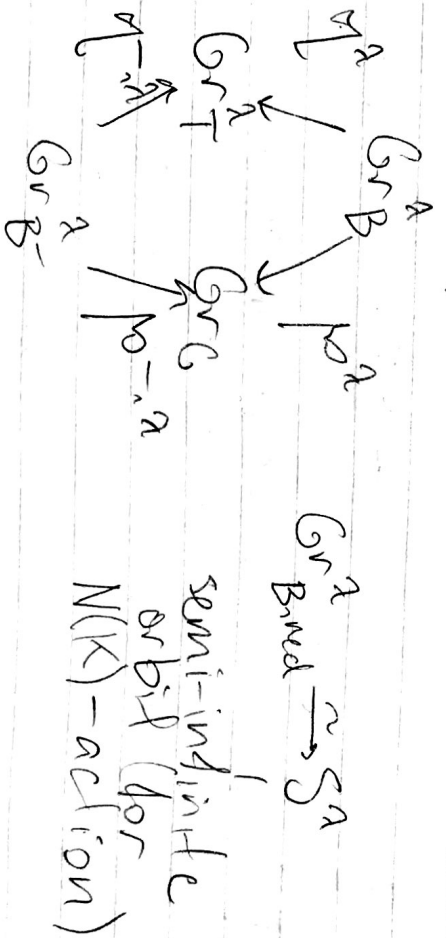
Construction of the fiber functor:

$$\text{Spn}_G \xrightarrow{\text{Hil}(\text{Gr}_G, -)} \text{D}(\text{Gr}_G) \xrightarrow{\text{Hil}(\text{Gr}_G, -)} \text{Vect}$$

First we will prove that this functor is  $\mathbb{Z}$ -exact.   
 $\mathbb{Z}$ -exact  $\Rightarrow$  go that it induces an exact functor   
up to some shifts

$$\text{Spn}_G^{\mathbb{Z}} \xrightarrow{\text{Hil}(\text{Gr}_G, -)} \text{Vect}^{\mathbb{Z}}$$

Consider the correspondences



Proposition There is a canonical isomorphism

$$(\rho^A)^* (\rho^A)^! \xrightarrow{\sim} (\rho^{-A})_! (\rho^{-A})^* \quad (*)$$

of functors

$$\text{D}(\text{Gr}_G)^{\text{T-mon.}} \xrightarrow{\sim} \text{D}(\text{Gr}_T^A) = \text{Vect}.$$

This is a standard application of Breen's theorem on  $G_m$ -actions.

One shows using dim. estimates for the  $S^1 \times \text{Gr}_n^*$  that

$$(n^2) * (n^{2k}) : \text{Sph}_n \rightarrow \text{D}(\text{Gr}_n) \text{ T-mon} \rightarrow \text{Vect.}$$

is left  $t$ -exact up to shift by  $\langle n, 2p \rangle$ . Verdier dual considerations show that  $(n^{2k})_{(1-p-n)}$  is right  $t$ -exact on  $\text{Sph}_n$  up to shift by  $\langle n, 2p \rangle$  which by (\*) would imply that both functors are  $t$ -exact up to shift.

Proposition There are canonical isomorphisms for  $\tilde{\tau}$  in  $\text{Sph}_n$

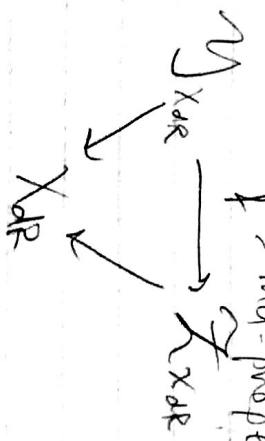
$$\begin{aligned} H_{\text{DR}}(\text{Gr}_n, \tilde{\tau}) &\xrightarrow{\sim} \bigoplus_{\alpha \in \mathbb{Z}} H^*(S^{2n-\alpha}, (\tilde{\tau}^{-n-\alpha})^* \tilde{\tau}) \\ &\xrightarrow{\sim} \bigoplus_{\alpha \in \mathbb{Z}} H^*(S^{-n}, (\tilde{\tau}^{-n-\alpha})^* \tilde{\tau}) \end{aligned}$$

Proof The isomorphism (\*) implies that the Cousin filtrations on  $H^*(\text{Gr}_n, \tilde{\tau})$  induced by the  $S^1$  and the  $S^{-1}$  split each other.  $\square$

In particular we obtain from this an exact, continuous fiber functor

$$F : \text{Sph}_{\text{Gr}_n} \rightarrow \text{Rep}(\tilde{\tau})^{\text{oblv}} \xrightarrow{\sim} \text{Vect}^{\text{oblv}}$$

For the symmetric monoidal structure on  $F$ :





Proposition There is an induced functor of factorization cat.'s

$$k_*: \text{IndCat}(\mathcal{Y}) \rightarrow \text{IndCat}(\mathcal{X}).$$

It follows that  $\text{Mor}(G_{\text{gr}}, -)$  defines an  $\mathbb{E}_2$ -monoidal functor

$$\text{Sph}_{G_{\text{gr}}, x} \rightarrow \text{Rep}(\tilde{T}).$$

hence an  $\mathbb{E}_2$ -monoidal functor

$$\text{Sph}_{G_{\text{gr}}, x} \rightarrow \text{Rep}(\tilde{T})^{\otimes \mathbb{E}_2}$$

after appropriate shifts. Since

$$\text{Fun}_{\mathbb{E}_2\text{-cat}}(\mathcal{L}, \mathcal{D}) \xrightarrow{\sim} \text{Fun}_{\mathbb{E}_2\text{-cat}}(\mathcal{L}, \mathcal{D})$$

for symmetric monoidal  $(1, 1)$ -categories  $\mathcal{L}$  and  $\mathcal{D}$ , we obtain the desired symm. mon. structure.

Finally, we show that  $F$  is conservative. Note that if  $\mathcal{T}$  in  $\text{D}(G_{\text{gr}})$  is such that  $(\mathfrak{a}, \mathcal{T})$  for all  $\mathfrak{a} \in \Lambda$ , then  $\mathcal{T} = \mathcal{D}$ .

Lemma Let  $F: \mathcal{C}_0 \rightarrow \mathcal{D}_0$  be an exact conservative functor between small abelian categories. Then  $\text{Ind}(F): \text{Ind}(\mathcal{C}_0) \rightarrow \text{Ind}(\mathcal{D}_0)$  is conservative.

NB: This lemma fails for DG categories.

As a corollary, it suffices to prove

that our fibered preservers compact objects,  
 and is conservative on compact objects.

Proposition The compact objects in  $\text{Sph}_{G_n^x}$  are precisely the holonomic complexes.

In particular, a compact object in  $\text{Sph}_{G_n^x}$  is supported on a finite-dimensional subspace of  $\mathbb{A}^n_x$ . (this is easier than the Prop.).

Let  $\mathcal{M}$  be a compact object of  $\text{Sph}_{G_n^x}$ . Then  $\mathcal{F}(\mathcal{M})$  is finite dimensional, because  $\mathcal{M}$  is supported on a finite-dim, subscheme of  $G_n$ .

If  $\mathcal{M} \neq 0$ , then there exists  $\lambda \in \mathbb{A}^n_+$  such that  $G_n^\lambda$  is an open in the support of  $\mathcal{M}$ . Then  $S_n \cap G_n^\lambda = S_n \cap G_n^\lambda$  is open in  $G_n^\lambda$  and closed in  $S_n$ . Moreover,  $\mathcal{M}|_{G_n^\lambda}$  is constant on  $G_n^\lambda$ , hence on  $S_n \cap G_n^\lambda$ . It follows that

$$H_{\text{br}}(S_n \cap G_n^\lambda; \mathcal{M}) \neq 0,$$

and we have

$$H_{\text{br}}^{< \lambda >}(S_n \cap G_n^\lambda; \mathcal{M}) \hookrightarrow \mathcal{F}(\mathcal{M}).$$