Basics of Category $\mathcal{O}$

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1 Introduction

This lecture is a dijest of chapter 1-3 of book [1]. We fix a complex semisimple lie algebra $\mathfrak{g}$, and a borel subalgebra $\mathfrak{b}$, with nilpotent radical $\mathfrak{n}$. So we have $\mathfrak{g} = \mathfrak{n}^{-1} \oplus \mathfrak{h} \oplus \mathfrak{n}$. Let $U(\mathfrak{g})$ be the universal enveloping algebra. By Poincaré-Birkoff-Witt (PBW), we have $U(\mathfrak{g}) = U(\mathfrak{n}^{-1})U(\mathfrak{h})U(\mathfrak{n})$. The category of $U(\mathfrak{g})$ modules is denoted $\mathfrak{g}$-mod, and the subcategory of finite dimension modules $\mathfrak{g}\text{-mod}^{f.d.}$. And the category of weight modules is denoted $\mathfrak{g}\text{-mod}^{\mathfrak{h}-s.s.}$ (for $\mathfrak{h}$-semisimple). If in addition the each weight space has finite dimension, the subcategory is denoted $\mathfrak{g}\text{-mod}^{\mathfrak{h}-s.s.,f.d.}$.

Before defining the Berstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$, we mention that is contains finite dimensional modules and highest weight modules (which contains Verma modules $M(\lambda)$). Each $M(\lambda)$ has a unique simple quotient called $L(\lambda)$. They are all the simple objects of $\mathcal{O}$.

Unlike $\mathfrak{g}\text{-mod}^{f.d.}$ which is semisimple by Weyl Reducibility Theorem, $\mathcal{O}$ is not semisimple. In such cases, we introduce the notion of *blocks*. We have decomposition

$$\mathcal{O} = \bigoplus_{\chi \in \mathrm{Spec}(Z(\mathfrak{g}))} \mathcal{O}_\chi. \tag{1}$$

Here $\mathrm{Spec} Z(\mathfrak{g})$ is just a pretentious way to write characters $Z(\mathfrak{g}) \to \mathbb{C}$, where $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$. We have an explicit description of $Z(\mathfrak{g})$ due to the Harish-Chandra isomorphism: $\xi : Z(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{h})^W$. $\mathcal{O}_\chi$ is sometimes called blocks.

Another decomposition result holds for projectives. The category $\mathcal{O}$ has enough projectives (and injectives). For each $\lambda \in \mathfrak{h}^*$, we associate one indecomposible projective $P(\lambda) \twoheadrightarrow L(\lambda)$. It turns out that any projective module is a direct sum of some $P(\lambda)$.

Filtration is yet another way to approximate a decomposition. For an object $M \in \mathcal{O}$, we have a finite length filtration $0 \subset M_1 \subset M_2 \cdots \subset M_n = M$, such that each $M_i/M_{i-1} \cong L(\lambda)$ for some λ. The multiplicity of $L(\lambda)$ is denoted $[M : L(\lambda)]$. We also have the notion of *standard filtration* (or Verma Flag), by requiring $M_i/M_{i-1} \cong M(\lambda)$ for some λ. The multiplicity of $M(\lambda)$ is denoted $(M : M(\lambda))$. Not all object in $\mathcal{O}$ has standard filtration. For example, $L(\lambda)$ usually don't have standard filtration.

One reason we consider standard filtration is that $M(\lambda)$ also form a basis of $K(\mathcal{O})$. In fact, we have $[M(\lambda)] = [L(\lambda)] + \sum_{\mu \leq \lambda} a(\lambda, \mu)[L(\mu)]$, where $a(\lambda, \mu) = [M(\lambda) : L(\mu)]$. So the change of basis is an "upper triangular" matrix, with diagonals all 1. The inverse relations can be written as $[L(\lambda)] = [M(\lambda)] + \sum_{\mu \leq \lambda} b(\lambda, \mu)[M(\lambda)]$. The coefficient $b(\lambda, \mu)$ is determined by Kazhdan-Lusztig conjecture.

Finally we have the fundamental result BGG reciprocity:

$$(P(\lambda), M(\mu)) = [M(\mu), L(\lambda)]. \tag{2}$$

2 Modules in Category $\mathcal{O}$

2.1 Definition of $\mathcal{O}$

Now we define the category $\mathcal{O}$ as a full subcategory of $\mathfrak{g}$-mod with objects M satisifying:

1. M is finitely generated;
2. M is a weight module, in other words it has a decomposition $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$.
3. M is locally $\mathfrak{n}$ finite: for each $v \in M$, the vector space $U(\mathfrak{n})v$ is finite dimensional.

There are several direct consequences of the axiom:

(P1) $\mathcal{O}$ is noetherian and abelian;

(P2) For each $M \in \mathcal{O}$, the set of weights appeared is contained in $\bigsqcup_{\lambda in I} \lambda - \Gamma$, where I is finite and Γ is the semigroup generated by positive roots (Φ^+).

Proof. Let V be the span of weight vector generators. Then $\dim V < \infty$. Consider $W = U(\mathfrak{b}) \cdot V$ and use PBW. □

(P3) If $L \in \mathfrak{g}\text{-mod}^{f.d.}$, and $M \in \mathcal{O}$, then $L \otimes M \in \mathcal{O}$. Furthermore, the functor $\mathcal{O} \to \mathcal{O} : M \mapsto L \otimes M$ is exact.

Proof. Suppose L has basis weight vectors $v_1, \cdots, v_n$, and M is generated by weight vectors $m_1, \cdots, m_l$, then we claim that $v_i \otimes m_j$ are weight vector generators of $L \otimes M$. In fact, let N be the module they generate. Then clearly $v \otimes m_i \in N$. Now if $x \in U(\mathfrak{g})$, we have $x(v \otimes m_j) = (xv) \otimes m_j + v \otimes (xm_j)$. So $v \otimes (xm_j) \in N$. □

(P4) L is locally $Z(\mathfrak{g})$ finite.

2.2 Highest Weight Modules and Verma Modules

Let $M \in \mathfrak{g}$-mod, we say $v \in M$ is a *maximal* vector if it is a weight vector and $\mathfrak{n} \cdot v = 0$. By assumption each object in $\mathcal{O}$ has a maximal vector. If $M = U(\mathfrak{g}) \cdot v$ for a maximal vector v, with weight λ, then we say that M is a highest weight module of weight λ.

Proposition 2.1. If M is a highest weight module, then it has a unique maximal submodule and a unique simple quotient.

For a weight $\lambda \in \mathfrak{h}^*$, we have the action of $\mathfrak{b}$ given by $\mathfrak{b} \twoheadrightarrow \mathfrak{h} \to \mathbb{C}$. Let $\mathbb{C}_\lambda$ denote the one dimensional $U(\mathfrak{b})$ module. We define the Verma module $V(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$. By the above above proposition, we write $N(\lambda)$ as the maximal submodule of $M(\lambda)$, and $L(\lambda)$ the unique simple quotient.

Proposition 2.2. $\dim \operatorname{Hom}_{\mathcal{O}}(L(\lambda), L(\mu)) = \delta_{\lambda\mu}$.

Proposition 2.3. $M(\lambda)$ is universal among highest weight modules of weight λ, and $L(\lambda)$ are all the simple objects of M.

Example 2.1. Consider $\mathfrak{sl}(2, \mathbb{C})$. $M(\lambda) = \bigoplus_{i \in \mathbb{N}} \mathbb{C} v_i$, where $hv_i = (\lambda - 2i)v_i$, $xv_i = (\lambda - i + 1)v_{i-1}$, $yv_i = (i+1)v_i$.

Proposition 2.4. $L(\lambda)$ is finite dimensional if and only if $\lambda \in \Lambda^+$. In such a case, $\dim L(\lambda)_\mu = \dim L(\lambda)_{w\mu}$ for $w \in W$, the Weyl group.

Proposition 2.5. If α is a simple root, and $n = \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}^+$, then $M(\lambda)$ has a maximal vector $v' = y_i^{n+1} v$, with weight $\mu = \lambda - (n+1)\alpha$.

Proof. Calculate $[x_i, y_i^{n+1}] = -(n+1)y_i^n(n - h_i)$. □

Now consider the action $s_\alpha \lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$. We define the shifted action (or dot action) $s_\alpha \lambda = \lambda - (n+1)\alpha = \lambda - \langle \lambda + \rho, \alpha^\vee \rangle \alpha$. In other words, $w \cdot \lambda := w(\lambda + \rho) - \rho$. We say that μ is *linked* to λ if there is a $w \in W$ such that $\mu = w \cdot \lambda$. We say that λ is *dominant* if there is no $\mu \leq \lambda$ that is also linked to λ. We have a similar notion of *antidominance*. Note that this dominance is different from the usual one. In particular, $\lambda \in \Lambda^+ - \rho$ is dominant.

It turns out that any linkage class has a unique dominant and a unique antidominant weight, and we have

Proposition 2.6. $M(\lambda) = L(\lambda)$ if λ is antidominant. $M(\lambda)(= P(\lambda))$ is projective if λ is dominant.

3 Decomposition with respect to the action of center

Recall that $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$. Let $z \in Z(\mathfrak{g})$, and $v \in M(\lambda)$ is the highest weight vector. Then $M_\lambda = \mathbb{C}v$. We have $h(zv) = z(hv) = \lambda(h)zv$. So $zv \in M_\lambda$, and therefore there exists $\chi_\lambda(z) \in \mathbb{C}$ such that $zv = \chi_\lambda(z)v$. Now we have obtained a character $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$.

The character χ_λ can be described explicitly. Let $\mathrm{pr} : U(\mathfrak{g}) \to U(\mathfrak{h})$ be the projection to $U(\mathfrak{h})$ by sending $x_i, y_i \mapsto 0$. λ extends to an algebra map $U(\mathfrak{h}) \to \mathbb{C}$, we have $\chi = \lambda \circ \mathrm{pr}$.

Although pr is not an algebra map, its restriction to $Z(\mathfrak{g})$ $\xi : Z(\mathfrak{g}) \to U(\mathfrak{h})$ is indeed a homomorphism, because $\bigcap_{\lambda \in \mathfrak{h}^*} \mathrm{Ker}\lambda = 0$.

Notice that we have nonzero map $M(w \cdot \lambda) \to M(\lambda)$, if $w \cdot \lambda \leq \lambda$. Using a density argument we have

Proposition 3.1. The image of map ξ is contained in $U(\mathfrak{h})^W$.

Theorem 3.1. (*Harish-Chandra*) The map $\chi : Z(\mathfrak{g}) \to U(\mathfrak{h})^W$ is an isomorphism.

We can use Harish-Chandra isomorphism to prove the following fact:

Proposition 3.2. $\mathcal{O}$ is Artinian. Moreover, if $M, N \in \mathcal{O}$, then $\dim \mathrm{Hom}_{\mathcal{O}}(M, N) < \infty$.

Proof. Notice that each $M \in \mathcal{O}$ has a filtration with successive quotients isomorphic to some highest weight module. Therefore it suffice to show that $M(\lambda)$ is artinian. Let $V = \sum_{w \in W} M(\lambda)_{w \cdot \lambda}$. If $N \supset N'$ are submodules of M, then $Z(\mathfrak{g})$ acts on N/N' with character χ_λ. So N/N' contains a maximal vector with weight μ such that $\chi_\mu = \chi_\lambda$. Therefore $\dim(N \cap V) > \dim(N' \cap V)$. □

For $M \in \mathcal{O}$, define

$$M^\chi = \{v \in M | \text{there exists } n \text{ depending on } z \text{ such that } (z - \chi(z))^n v = 0, \text{ for all } z \in Z(\mathfrak{g})\}. \tag{3}$$

Then it is clear that $M = \bigoplus_\chi M^\chi$. Let $\mathcal{O}_\chi$ be the subcategory with objects $M = M^\chi$. Therefore we have

Proposition 3.3. $\mathcal{O} = \bigoplus_\chi \mathcal{O}_\chi$.

A brief explanation of *blocks*. If A, B are simple objects, and has non-split extension, $0 \to A \to M \to B \to 0$, we put A, B in a same block. If there is a sequence of simple objects $A = A_0, A_1, \cdots, A_n = B$, such that each adjacent pair is in the same block, then we put A, B in the same block. If all the simple factors of M is in the same block, we put M in that block too. It turns out $\mathcal{O}_\chi$ is a block if $\chi = \chi_\lambda$, and $\lambda \in \Lambda$. If $\chi \in \mathfrak{h}^*$, then it is possible that $\mathcal{O}_{\chi_\lambda}$ can be decomposed into abelian subcategories.

4 Characters

Recall that if M is a finite dimensional module, we define ch M as an element in the group ring $\mathbb{Z}\Lambda$: ch $M = \sum_\lambda \dim(M_\lambda)e(\lambda)$. Here $e(\lambda)$ represents a generator in $\mathbb{Z}\Lambda$. We have $e(\lambda)e(\mu) = e(\lambda + \mu)$.

For $\mathcal{O}$, or more generally $\mathfrak{g}\text{-mod}^{\mathfrak{h}-s.s.,f.d.}$, we define ch M as a function $f : \mathfrak{h}^* \to \mathbb{Z}$, with $f(\lambda) = \dim M_\lambda$. Then the product now becomes convolution. In fact, we have characteristic function $e(\lambda)$, which values 1 on λ and 0 else where. Intuitively, we think of $e(\lambda)$ as exponentials and f as a fourier-transform $\sum_\lambda f(\lambda)e(\lambda)$.

Proposition 4.1. ch $(L \otimes N) =$ ch $L *$ ch N, if $L \in \mathfrak{g}\text{-mod}^{f.d.}$. If we have short exact sequence $0 \to M' \to M \to M'' \to 0$, then ch $M =$ ch $M' +$ ch M''.

The character of Verma modules are simple, and they are described using a single function p. $p(\gamma)$ is defined by the number of solutions to $\gamma = -\sum_{\gamma_\alpha} \alpha$.

Proposition 4.2. ch $M(0) = p$, ch $M(\lambda) = e(\lambda) * p$.

The character of $L(\lambda)$ is much more complicated. By section 2, 3, we can write ch $M(\lambda) = \sum a(\lambda, \mu)$ch $L(\mu)$, where $\mu \leq \lambda$ and μ linked to λ. $a(\lambda, \lambda) = 1$. Inverting these relations we have

$$\text{ch } L(\lambda) = \sum_{w \in W, w\cdot\lambda \leq \lambda} b(\lambda, w)\text{ch } M(w \cdot \lambda). \tag{4}$$

The coefficients $b(\lambda, w)$ is in general difficult to compute, and is given by Kazhan-Lusztig. Now we compute $b(\lambda, w)$ for $\lambda \in \Lambda^+$, which is equivalent to the Weyl character formula.

Theorem 4.1. For $\lambda \in \Lambda^+$, the coefficient $b(\lambda, w) = (-1)^{l(w)}$, where $l(w)$ is the length.

Before proving the theorem, we rewrite p. Let $f_\alpha(\lambda) = \sum_{k \in \mathbb{N}} e(\lambda - k\alpha)$. Then we have identities

$$p = \prod_{\alpha > 0} f_\alpha \tag{5}$$

$$(e(0) - e(-\alpha)) * f_\alpha = e(0). \tag{6}$$

The key to the proof is to introduce the Weyl denominator $q := \prod_{\alpha>0}(e(\alpha/2) - e(-\alpha/2))$. Then we have

$$q * \text{ch } M(\lambda) = e(\lambda + \rho). \tag{7}$$

Proof. of Theorem 4.1
We have:

$$q * \text{ch } L(\lambda) = \sum_{w \in W} b(\lambda, w)e(w(\lambda + \rho)). \tag{8}$$

Note the $w \cdot \lambda \leq \lambda$ for all $w \in W$ in this case. Consider the action of s_α, q is changed to $-q$, and we already know from Proposition 2.4 that ch $L(\lambda)$ is invariant under W. So we have $b(\lambda, w) = -b(\lambda, s_\alpha w)$. Since $b(\lambda, \lambda) = 1$, we have $b(\lambda, w) = (-1)^{l(w)}$. ☐

The above result suggests to write ch $L(\lambda)$ as a Euler characteristic. In other words, there is a BGG resolution:

$$\cdots \to \bigoplus_{w \in W, l(w)=k} M(w \cdot \lambda) \to \cdots \to M(\lambda) \to L(\lambda) \to 0. \tag{9}$$

5 Projectives and BGG Reciprocity

We first remark that $\mathrm{Ext}_{\mathcal{O}} \neq \mathrm{Ext}_{U(\mathfrak{g})}$. To define the derived functor of Hom we need enough projectives. We have mentioned that $M(\lambda)$ is projective if λ is dominant. To get more projectives, we use the following observation:

Proposition 5.1. If P is projective, and L fiinite dimensional, then $P \otimes L$ is projective.

Proof. We have $\mathrm{Hom}(P \otimes L, M) = \mathrm{Hom}(P, L^* \otimes M)$. This is compatible with $U(\mathfrak{g})$ structure. □

Proposition 5.2. If $M \in \mathfrak{g}\text{-mod}^{f.d.}$, then $M(\lambda) \otimes M$ has standard filtration with successive quotients $M(\lambda + \mu)$, with $(M(\lambda) \otimes M : M(\lambda + \mu)) = \dim M_\mu$.

Proof. Use the tensor identity

$$(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} L) \otimes M \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (L \otimes M). \tag{10}$$

Take $L = \mathbb{C}_\lambda$, and use (P3) of section 2. □

This proposition has two immediate consequences.

Proposition 5.3. $\mathcal{O}$ has enough projectives (and injectives).

Proof. For $L(\lambda)$, consider $M(\lambda + n\rho) \otimes L(n\rho)$. □

Therefore we can associate each $L(\lambda)$ a projective $P(\lambda) \twoheadrightarrow L(\lambda)$. If in addition we require $P(\lambda)$ is essential (it has no proper submodule maps onto $L(\lambda)$), the it is uniquely determined. Clearly $P(\lambda)$ is indecomposible. Moreover, using just the universal properties of $P(\lambda)$, we have

Proposition 5.4. Any projective in $\mathcal{O}$ is a direct sum of $P(\lambda)$.

Proposition 5.5. $P(\lambda)$ has standard filtration.

Proof. By proposition 5.2, we can embed $P(\lambda)$ as a direct summand of $M(\lambda + n\rho) \otimes L(n\rho)$ for large ρ. □

We are ready to prove the following fundamental result.

Theorem 5.1. *BGG Reciprocity.*

$$(P(\lambda) : M(\mu)) = [M(\mu), L(\lambda)] = [M(\mu)^\vee, L(\lambda)]. \tag{11}$$

Proof. The key is to identify both sides as $\dim \mathrm{Hom}_{\mathcal{O}}(P(\lambda), M(\mu)^\vee)$. □

Lemma 1. If M has standard filtration, then

$$\dim \mathrm{Hom}_{\mathcal{O}}(M, M(\mu)^\vee) = (M : M(\mu)). \tag{12}$$

Proof. Use induction on the length of M. Use $\mathrm{Ext}(M(\mu), M(\lambda)^\vee) = 0$, $\dim \mathrm{Hom}_{\mathcal{O}}(M(\mu), M(\lambda)^\vee) = \delta_{\lambda\mu}$. A map $M(\lambda) \to M(\lambda)^\vee$ is given by $M(\lambda) \twoheadrightarrow L(\lambda) \cong L^\vee \hookrightarrow M(\lambda)^\vee$. □

Lemma 2. For any $M \in \mathcal{O}$, $\dim \mathrm{Hom}_{\mathcal{O}}(P(\lambda), M) = [M : L(\lambda)]$.

Proof. Again use induction on the induction on the length, except we use the usual filtration of M. □

References

[1] James E. Humphrey: Representations of Semisimple Lie Algebra in the BGG Category $\mathcal{O}$. 2008

[2] Dennis Gaitsgory: Lecture notes on Geometric Represenation Theory. Fall 2005.