Rep theory in char 0 (reductive grps).

G reductive. e.g. SL_n.

B Borel

N nilpotent.

T maximal torus.

1) Each rep of G splits as $\oplus$ irreducible.

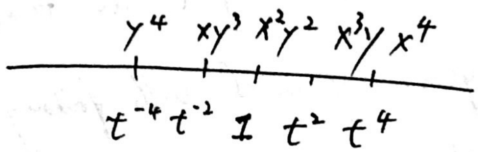
2) irreds are classified by highest weight.

Example: SL_2.

For $n \in \mathbb{Z}$, have rep of SL_2:

$\mathrm{Sym}^n V$ V standard rep.

If we look at eigenvalues under $T = \begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix}$.

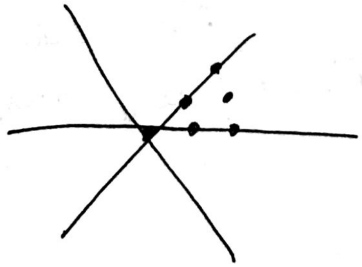


x, y: basis s.t. $\begin{bmatrix} t & \\ & 1/t \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} tx \\ 1/t\, y \end{bmatrix}$.

weight diagram.

Thm irreps $\Longleftrightarrow$ dominant weights.

SL_3

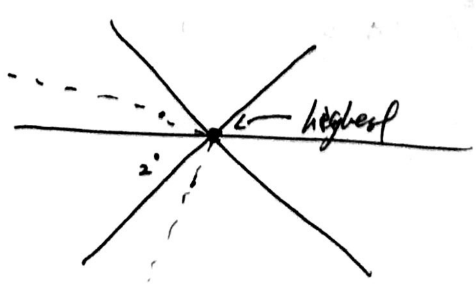


Second easiest: Cat $\mathcal{O}$.

Rep of $\mathfrak{g}$ Lie algebra. ($\mathfrak{g}$ semisimple)

For every weight λ, have

Verma: $V(\lambda) = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$.



$U(\mathfrak{n}^-) v \overset{\text{hwv.}}{\simeq} V(\lambda)$

irreps $:= L_\lambda$.

$\mathfrak{n}$ acts trivially on highest weight.

$\mathfrak{t}$ acts by weight $\mathfrak{n}^-$ acts freely.

BGG resolution.

$$\underset{\ell(w)=2}{\bigoplus} \cdots \to \underset{\ell(w)=1}{\bigoplus} V_{w\cdot\lambda} \to V_\lambda \to L_\lambda$$

where $w\cdot\lambda = w(\lambda+\rho)-\rho$.

thus $\mathrm{char}(L_\lambda) = \sum (-1)^{\ell(w)}[V_{w\cdot\lambda}] = \sum (-1)^{\ell(w)}$ (something easy).

Cat $\mathcal{O}$ is a natural setting to do this.

:= fin. gen. $\mathfrak{g}$-mod with following properties:

- $\mathfrak{n}$ acts locally nilpotently, i.e. $\forall a\in\mathfrak{n}$, $v\in V$. $a^k v = 0$ $\forall k>>0$.
- $\mathfrak{t}$ acts semisimply with integer weight eigenvalues.

Examples of objects: $\underset{\text{standard}}{V_\lambda}$, $\underset{\text{simple}}{L_\lambda}$, $\underset{\text{costandard}}{V_\lambda^\vee} := (U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^-)} \mathbb{C}_\lambda)^*$ ← contragredient dual. P_λ ← projective

Relations:

$$V_\lambda \to L_\lambda \to V_\lambda^\vee$$

Also have a projective cover $P_\lambda \to L_\lambda$. s.t. $\mathrm{Hom}(P_\lambda, M) \simeq \mathbb{C}^{[M:L_\lambda]}$ ← w/ Jordan-Hölder series

---

Axiomatization known as highest weight cat.

$$\begin{cases} V_\lambda \twoheadrightarrow L_\lambda \hookrightarrow V_\lambda^\vee \\ P_\lambda \text{ has filtration by } V_\lambda\text{'s} \\ \text{etc.} \end{cases}$$

Example: $[V_\lambda : L_{\lambda'}]$

- which $L_{\lambda'}$ show up?

$Z(U(\mathfrak{g})) \simeq U(\mathfrak{h})^W$. Thus $Z(U(\mathfrak{g}))$ acts by character on ~~everything~~ $V_\lambda, L_\lambda, V_\lambda^\vee, P_\lambda$.

Namely, the character which sends $P \in U(\mathfrak{h})^W$ to $P(\lambda+\rho)$

"Proof". $Z(U(\mathfrak{g}))$ acts by this char. on the highest weight of V_λ.

Thus by this char. on $U(\mathfrak{n}^-)v \dashrightarrow$ everything.

Corollary: $\mathcal{O} \cong \bigoplus_{\chi \in Z(U(\mathfrak{g}))} \mathcal{O}_\chi$ (because Hom = Ext = 0 for objects w/ different characters).

which L_λ appear in $\mathcal{O}_\chi$? Exactly the λ in one $(W,\cdot)$-orbit.

(Exercise).

Cor. $[V_\lambda : L_{\lambda'}] \neq 0$ only if λ, λ' in same block.

and ~~[illegible]~~ (some ~~[illegible]~~ superiority condition).

let λ_0 be the unique dominant weight in this orbit.

$\lambda = w \cdot \lambda_0$

$\lambda' = w' \cdot \lambda_0$ then $[V_\lambda : L_{\lambda'}] \neq 0$ iff. $w' \geq w$ in the Bruhat order.

(i.e. $w' \geq w$ iff. $\exists$ simple word rep. $w' = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_n}$, $w = s_{\alpha_{j_1}} s_{\alpha_{j_2}} \cdots s_{\alpha_{j_m}}$)

In fact, we completely understand these multiplicities.

Consider K_0(block).

V_λ, L_λ are bases. $[V_\lambda : L_{\lambda'}]$ are the entries in the $L \to V$ transition matrix.

define $[L_\lambda : V_{\lambda'}]$ is the entries in the $V \to L$ matrix.

(Can be negative.)

Thm $[L_{\lambda'} : V_\lambda] = (-1)^{l(w)-l(w')} \underbrace{P_{w,w'}}(1)$.

Kazhdan-Lusztig polynomials.

Cor. They have non-neg. integer coeff.

~~[illegible]~~ Note: the answer doesn't depend on which block this is.

Almost all blocks are equivalent.

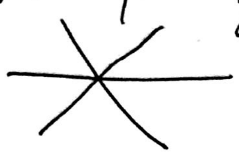
All blocks with $|W\text{-orbit}| = |W|$ are equivalent (related by translation functors).

i.e. $M \to (M \otimes L_{\lambda'-\lambda^0})|_{\text{block } \lambda'}$ if $\lambda' - \lambda^0$ dominant.

Have more general setting.

(This was for $\mathfrak{g}$ fin. dim.) $\rightsquigarrow$ (can also do for $\mathfrak{g}$ Kac-Moody).

e.g. $\mathfrak{gl}_3$: reflection system



$\rightsquigarrow$

d: dominant chamber

(type A_2)

in fact: almost everything discussed goes though in this setting.

W^{aff} is the symmetry group.

roughly : $\mathfrak{gl}_3((t))$.

again have Cat $\mathcal{O}$ etc.

Special about affine Weyl groups: they have large abelian subgrp of translations.

Thus we have two perspectives: 1) Kac-Moody

2) Translated copies ("affine")

Now, SL_2 in char p. (semisimplicity fails.)

Start with $V_n := \mathrm{Sym}^n(V)$. Is V_n still irreducible?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \curvearrowright x^n \rightsquigarrow (ax+by)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} x^k y^{n-k}$$

$\binom{n}{k}$ often zero.

Namely, $\binom{n}{k}$

$$\equiv \binom{n_i}{k_i}\binom{n_{i-1}}{k_{i-1}}\cdots\binom{n_0}{k_0} \bmod p.$$

$n \underset{(\text{base } p)}{\doteq} n_i\, n_{i-1} \cdots n_0$

$k \doteq k_i \cdots k_0$.

$\not\equiv 0$ iff each $n_a \geq k_a$.

i.e. can add k and $n-k$ in base p without carries.

Let's define this as $k \leq_p n$.

$(ax+by)^n = \sum_{k \le_p n} \cdots x^k y^{n-k}$.

irred. submodule given by span$(x^k y^{n-k}$ for $k \le_p n) =: L_n$.

dim $L_n = (n_i+1)(n_{i-1}+1)\cdots(n_0+1)$.

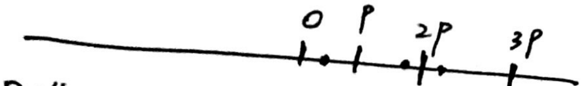
Thm: these are all the irred.

e.g. if n = power of p. dim $L_x = 2$.

Because rep. theory no longer semisimple.
Can start to ask refined questions.

e.g. 1) What are the blocks?

Given by dominant weights in the affine Weyl gp orbits.



"Pf": $Z(\mathcal{U}(\mathfrak{g})) \simeq (\mathrm{Sym}\, \mathfrak{h})^W$.

char p! In other words, λ and $\lambda + p\alpha$ have the same character.

(Thus we see affine Weyl showing up)

2) What are (co)standards?

In char 0, have a way of constructing V_λ.
Namely, $H^0(G/B, \mathcal{O}(\lambda))$. We know dimension of them by HRR + H^1 vanish by Kodaira vanishing.

Can do the same in char p.

$V_\lambda := H^0(G/B, \mathcal{O}(\lambda))$. Again have dim.
H^1 vanishing: Kempf vanishing.
(Hint: $Fr_* \mathcal{O}(-\rho) = \mathcal{O}(-\rho)^{p^{\dim G/B}}$)

3) again, can ask for $[V_\lambda : L_{\lambda'}]$.
When is it nonzero?

Take λ_0 dominant.
$\lambda = w.\lambda_0$
$\lambda' = w'.\lambda_0$ $(w, w' \in W_{aff})$

Answer: nonzero $w_0 w \underset{\text{Bruhat}}{\geq} w_0 w'$

w_0: largest element of finite Weyl.

Thm If $\lambda = p\alpha + \beta$ ~~β small exponent~~ $\alpha = \lfloor \frac{\lambda}{p} \rfloor$.
then $L_\lambda = L_\alpha^{(1)} \otimes L_\beta$. (Steinberg tensor product thm).

(1) ← Frob twist

(compare w/ dim L_n from before : $L_n = \underbrace{L_{n_1 0 0 \cdots 0}} \otimes L_{0 n_1 \cdots 0 0} \otimes \cdots \otimes L_{0 \cdots 0 n_0}$)

↑ looks essentially like $L_{n_i}^{(i)}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \curvearrowright L^{(1)} := \begin{bmatrix} a^p & b^p \\ c^p & d^p \end{bmatrix} \curvearrowright L.$$

Thus suffices to understand L_n for small n.

Def : $\lambda \in \mathrm{Box}_{\text{fundamental}}$ if $\lfloor \frac{\lambda}{p} \rfloor = 0$.

Claim : $[V_\lambda : L_{\lambda'}] = P_{w_0 \lambda', w_0 \lambda}(1)$ if $\lambda, \lambda' \in \mathrm{Box}_{\text{fundamental}}$.

[Lusztig's Conjecture]. (true for $p \gg 0$). (False for p small.)

Note: this is subtler than affine case.

History: Conj. by Lusztig (1979)

Proven by AJS (1990)

Reproven by Fiebig (2000s) ... double exponential bound

Reproven by Romen (2000s)

Before : $[L_\lambda : V_{\lambda'}] =$ Kazhdan-Lusztig Polynomial.

(Compare w/ above!) (note: affine ver. doesn't admit simple inversion formulae)

Schematic for proving that ~~doesn't~~ work :

step 1 : show $\simeq$ Cat $\mathcal{O}$ for affine Lie alg.

Let's say sth about proof.

Important picture: $G_1 \to G \xrightarrow{Fr} G$.

have in fact many interrelated rep theory.

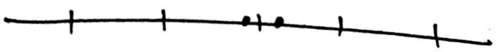
Rep G

Rep G_1

Rep G_1T

- Thm: the irred for G_1 (finite grp scheme) are the L_λ, $\lambda \in$ fundamental box
- Rep of G_1 = restricted rep of $\mathfrak{g}$.

In practice, we study Rep G_1T. aka weight-graded G_1-rep.



Blocks are now W_{aff}-orbits.
actual

have periodized everything.

What are standards now?

e.g. $\mathrm{Ind}_{B_1T}^{G_1T} \mathbb{C}_\lambda =: V_\lambda$ Exercise: $\dim V_\lambda = p^{\dim G/B}$.

(aka Baby Verma modules.)

Step 1: Recasting Lusztig's Conjecture:

$$[L_\lambda : V_{\lambda'}] = Q_{\lambda',\lambda}(1)$$

periodic KL-polynomial. (or something)

KL poly $\longleftrightarrow$ aff flag

$\Updownarrow$

periodic KL poly $\longleftrightarrow$ semi-infinite flag

Step 2: switch to quantum grp.

Step 3: prove the statement there (AJS).

Remark: $D(\mathfrak{g}\text{-mod}) \simeq DCoh(T^*B)$. ← note: also doesn't match with the $\frac{\infty}{2}$-t-structure.

t-structure ⇝ t-structure. } not same t-structure.
independent of p.

naive: $D^G Coh(T^*B) \simeq Whit(Fl)$ This doesnot work.

BR: reduce to positivity results.

better: relate these t-structures.

The flip comes then from Koszul duality. (Projective) ⇝ (Tilting) $\xrightarrow{\text{Koszul}}$ (Simples).

[baby Verma : simple]

⇕

Hom(Projective, baby Verma)

⇕

[Proj ; Wakimoto] ⇜ Hom(Wakimoto, baby Verma)
easy

⇕ change of t-structure.

[Tilting : Wakimoto]

⇕ Koszul Duality.

[Simple : standard].