Local Langlands Conjecture for GL_2

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Contents

1 Lecture 1

1.1 Quasi-Characters of Profinite Groups

Let G be a topological (Hausdorff) group. We say that G is *profinite* if it is totally disconnected and compact. Equivalently, G is the inverse limit of finite groups. Then, G has a basis of neighbourhoods at identity given by finite index open normal subgroups. We say that G is *locally profinite* if G is locally compact and totally disconnected. Then, any compact subgroup of G would be profinite.

Theorem 1.1. *Let G be a topological group and H, H_1, H_2 be topological subgroups. The following are true.*

a) H is open if and only if G/H is discrete. If H is open, it is closed.

b) If H is finite index and closed, it is open.

c) The map $G \to G/H$ is open.

d) If H is compact, the map $G \to G/H$ is closed.

e) G/H is Hausdorff if and only if H is closed.

f) If G is profinite, H is compact open, then G/H is finite index.

g) (van Dantzig) If G is locally profinite, compact open subgroups form a basis of neighbourhoods of identity and compact subgroups are contained in open compact subgroups.

Example 1.2. $\mathbb{Q}_p$, $\mathbb{Q}_p^\times, \mathbb{A}_{\mathbb{Q}}^\infty, (\mathbb{A}_{\mathbb{Q}}^\infty)^\times, \mathrm{GL}_n(\mathbb{Q}_p)$ are locally profinite while $\mathbb{Z}_p$, $\mathbb{Z}_p^\times, \widehat{\mathbb{Z}}, \mathrm{GL}_n(\widehat{\mathbb{Z}})$ are profinite. Similarly for local fields.

Let F be a local field, $\mathcal{O}_F$ its ring of integers and V a vector space of dimension n over F. An $\mathcal{O}_F$-lattice L in V is a finitely generated $\mathcal{O}_F$ submodule such that $L \otimes_{\mathcal{O}_F} F = V$.

Proposition 1.3. *If L is a $\mathcal{O}_F$-lattice in V, then L is free of rank n over $\mathcal{O}_F$.*

Proof. By minimality. □

Remark 1.4. A lattice for a non-archimedean local field is not discrete.

Definition 1.5. Let G be a locally profinite group. A continuous homormophism $\psi : G \to \mathbb{C}^\times$ is said to be a *quasi-character*. If the image lies in $S^1 \subset \mathbb{C}^\times$, it is said to be a *unitary-character*.

Proposition 1.6. *A homomorphism $\psi : G \to \mathbb{C}^\times$ of abstract groups is continuous if and only if $\ker \psi$ is open in G. Moreover, if G is a union of compact open subgroups, then ψ is necessarily unitary.*

Proof. By topology, this follows. Choose an open neighbourhood around $1 \in S^1$, and look at preimage etc. □

Example 1.7. If F is a local field, all quasi-characters $F \to \mathbb{C}^\times$ are unitary.

Example 1.8. Let ω be a uniformizer. Then, $F^\times \cong \mathcal{O}_F^\times \times \omega^{\mathbb{Z}}$. Thus, a (quasi)-character of $F^\times$ is a character of $\mathcal{O}_F^\times$ which is unitary by compactness, and a number associated to $\omega \in \omega^{\mathbb{Z}}$. For instance, $|\cdot| : F^\times \to \mathbb{C}^\times$ is a non-unitary character.

1.1.1 Duality Theorems

Let G be an abelian topological group which is locally compact. We define $\widehat{G}$ to be unitary/Pontryagin dual of G i.e. the group of unitary characters of G under the compact open topology. We have the following general statement.

Proposition 1.9 (Pontryagin duality). *The pairing $G \times \widehat{G} \to S^1$ induces an isomorphism $G \to \widehat{\widehat{G}}$.*

Additive Characters Let F be a local field. Then all characters of F are unitary.

Definition 1.10. Let $\psi \in \widehat{F}, \psi \neq 1$. The *conductor* or *level* of ψ is the smallest c such that $(\omega^c) \in \ker \psi$.

Proposition 1.11 (Additve Pontryagin Duality). *Let $a \in F$ and $\psi \in \widehat{F}$, $\psi \neq 1$ of conductor c.*

1) *The map $a\psi : x \mapsto \psi(ax)$ is a character of F. If $a \neq 0$, the character $a\psi$ has conductor $c - v_f(a)$.*
2) *The pairing $F \times F \to S^1$ given by $(a, x) \mapsto \psi(ax) = a\psi(x)$ induces an isomorphism $F \cong \widehat{F}$.*

Proof. Boring. □

Example 1.12. How do we produce one non-trivial character for local fields F? For $\mathbb{Q}_p$, take $\psi_p : \mathfrak{Q}_p \to S^1$ given by $\psi_0(\alpha) = e^{2\pi i(\alpha \mod \mathbb{Z}_p)}$. This gives an isomorphism

$$\mathbb{Q}_p/\mathbb{Z}_p \cong \mathbb{Z}(p^\infty)$$

, of topological groups, where $Z(p^\infty)$ is the (discrete) Prufer p-group. For finite extensions $F/\mathbb{Q}_p$, we can take $\psi_F = \psi_0 \circ \mathrm{Tr}_{F/\mathbb{Q}}$.

1.1.2 Multiplicative Characters

Let's look at $F^\times$. We notice that $\mathcal{O}_F^\times$ is the unique maximal compact open subgroup of F, in sharp contrast with F.

Definition 1.13. Let $\psi : F^\times \to \mathbb{C}^\times$ be a character. The *conductor* or *level* of ψ is defined to be the smallest c such that $1 + \omega^c\mathcal{O}_F \in \ker \psi$.

Since $\mathcal{O}_F^\times/(1+\omega^c\mathcal{O}_F)$ is a finite abelian group, the characters of $F^\times$ can be determined by the corresponding characters of finite groups.

1.2 Smooth Representations and Induction

Let G be a locally profinite group and (π, V) be a representation of G.

Definition 1.14. We say that a representation (π, V) of G is smooth if for all $v \in V$, there is a compact open $K_v \subset G$ such that $\pi(x)v = v$ for all $x \in K_v$. If V is smooth, U is a G-stable subspace, then U and V/U are smooth. We say that (π, V) is *admissible* if the space V^K is finite dimensional for all compact open subgroups of G. We say that (π, V) is *irreducible* if V has no G-stable subspace U. A *homomorphism* $(\pi_1, V_1) \to (\pi_2, V_2)$ is a linear map $f : V_1 \to V_2$ such that it twists the π_i. Then, the class of smooth representations of G forms an abelian category $\mathrm{Rep}(G)$.

Example 1.15. A 1-dimensional representation is smooth if and only if it is given by a continuous character.

Example 1.16. If G is profinite, and (π, V) is smooth and irreducible, then V is finite dimensional. Indeed, $V = \bigcup_K V^K$ where K is compact open. Each $v \in V$ non-zero lies in some V^K, and $\{\pi(g)v : g \in G/K\}$ spans V. We know that $(G : K)$ is finite and $K' = \bigcap_{g \in G/K} gKg^{-1}$ is open normal which acts trivially on V. So, V is irreducible over G/K' which is finite, and hence is finite dimensional.

We can prove that the usual theorem of V being a sum of irreducible G-subspace is equivalent to it being a direct sum, which is equivalent to showing that every G-stable subspace of V has a G-complement. So, are representations of G-semisimple?

Lemma 1.17. *If G is a locally profinite group and (π, V) be a smooth representation of G. If K is a compact open subgroup of G, V is a direct sum of irreducible K-subspaces.*

Proof. Each $v \in V$ is stabilized by some open normal subgroup K' of K and the K-span W of v is finite dimensional on which K' acts trivially. So, W is a representation of K/K', and so is a sum of irreducible K subspaces. Now note that $v \in V$ was arbitrary. □

If G is a locally profinite group, K compact open, then let $\widehat{K}$ denote the set of equivalence classes of irreducible smooth representations of K. For each $\rho \in \widehat{K}$, we can talk about the ρ-isotypic component of a smooth representation (π, V) of G i.e. sum of all irreducible K-subspaces of V of class ρ.

Proposition 1.18. *Let (π, V) be smooth over G and K compact open.*

1) *V is a direct sum of its K-isotypic components.*
2) *Taking isotypic components behaves well under homomorphisms i.e. if $f : V \to W$ is in $\mathrm{Rep}(G)$, then*

$$f(V^\rho) \subset W^\rho \text{ and } W^\rho \cap f(V) = f(V^\rho)$$

Corollary 1.19. *Let $U \to V \to W$ be a sequence of G-spaces. Then, it is exact if and only if*

$$U^K \to V^K \to W^K$$

is exact.

Proof. By the well-behavedness of taking isotypic component, we can check the conditions of exactness on K-invariant subspaces. □

Corollary 1.20. *Let K be a compact open subgroup of a locally profinite group G and (π, V) a smooth representation of G. Let $V(K)$ be the linear span of*

$$\{v - \pi(k)v \,|\, v \in V, h \in H\}.$$

Then, with

$$V(K) = \bigoplus_{\rho \in \widehat{K}, \rho \neq 1} V^\rho$$

and $V(K)$ is the unique K-complement of V^K in V.

Proof. The sum $W = V^\rho$ with ρ non-trivial is a K-complement of V^Kin V. So, we have an exact sequence

$$0 \to W \to V \to V^K \to 0$$

and $V(K)$ is contained in any K-map $V \to V^K$. So, $W \supseteq V(K)$. However, if U is irreducible over K and in the class of $\rho \neq 1$, $U(K) = U$, since U is irreducible and U is not trivial. Since this is true for non-trivial isotypic component $V^\rho = V^\rho(K) \subset V(K)$.

□

1.2.1 Induction

Definition 1.21. Let G be a locally profinite group, and H a closed subgroup. Let (σ, W) be a smooth representation of H. Let X be the $\mathbb{C}$-vector space of functions $f : G \to W$ such that

1) $f(hg) = \sigma(h)f(g)$ for all $h \in H$, $g \in G$ i.e. it's σ-semilinear.
2) there is a compact open subgroup K_f of G such that $f(gx) = f(g)$, so f is constant on gK for all $g \in G$.

We let $\Sigma : G \to \mathrm{Aut}_{\mathbb{C}}(X)$ to be the map

$$\Sigma(g)(f) : x \mapsto f(xg) \qquad \text{for } g, x \in G.$$

The pair (Σ, X) provides a smooth representation of G, with smoothness holding because of 2). We write

$$(\Sigma, X) = \mathrm{Ind}_H^G \sigma$$

So, we have a functor $\mathrm{Ind}_H^G : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$. There is a canonical H-homomorphism

$$\alpha_\sigma : \mathrm{Ind}_H^G \sigma \to W$$

$$f \mapsto f(1),$$

which is indeed a H-homomorphism because of 1). We also have a functor $\mathrm{Res}_H^G : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$ which is the stupid functor.

Proposition 1.22 (Frobenius Reciprocity)**.** *Let G be locally profinite, H a closed subgroup, (π, V) a smooth representation of G and (σ, W) a smooth representation of H. Then, we have a functorial bijeciton*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \to \mathrm{Hom}_H(\pi, \sigma)$$

$$\phi \mapsto \alpha_\sigma \circ \phi$$

So, basically,

$$\mathrm{Ind}_H^G \dashv \mathrm{Res}_H^G.$$

Proof. We can construct an explicit inverse for a H-homomoprhism $f : V \to W$. Let $f_* : V \to \mathrm{Ind}\,\sigma$ be such that for $v \in V$, $f_*(v) = g \mapsto g(\pi(g)v))$. Then, $f \mapsto f_*$ is the inverse. □

Proposition 1.23. *The functor Ind_H^G is additive and exact.*

Proof. Too lazy to read it. □

1.2.2 Compact Induction

Let H, which an open subgroup of a locally profinite group G. Then, H is closed and is thus locally profinite. Let (σ, W) be a smooth representation of H. We can consider X_c to be the space of *compactly supported functions modulo* H i.e. functions $f \in X$ such that the image of supp$f$ in $H\ G$ is compact. The space X_c is stable under the action of G and provides another smooth representation of G. So, we get a functor

$$c\text{-}\operatorname{Ind}_H^G : \operatorname{Rep}(H) \to \operatorname{Rep}(G)$$

which we refer to as *compact induction*. It is additive and exact, and we have a canonical G-embedding

$$c\text{-}\operatorname{Ind}_H^G \sigma \to \operatorname{Ind}_H^G \sigma$$

because $X_c \subset X$. This provides an equivalence of categories if and only if G/H is compact.

Again, we have a H-homomophism

$$\alpha_c^\sigma : W \to c\text{-}\operatorname{Ind}_H^G \sigma$$
$$w \mapsto f_w$$

where $f_w \in X_c$ is supported in H and $f_w(h) = \sigma(h)w$ for $h \in H$. If H is open, the map $\alpha_\sigma^c : W \to c\text{-}\operatorname{Ind}_H^G$ is a H-isomorphism onto those functions $f \in X_c$ such that supp$f \subset H$.

Proposition 1.24. *Let G be a locally profinite group, H a closed subgroup. The compact induction functor is exact and addtive. If moreover, H is open, then*

$$\operatorname{Hom}_G(c\text{-}\operatorname{Ind}\sigma, \pi) \to \operatorname{Hom}_H(\sigma, \pi)$$
$$f \mapsto f \circ \alpha_\sigma^c$$

is a functorial isomorphism. Thus,

$$c\text{-}\operatorname{Ind}_H^G \vdash \operatorname{Res}_H^G .$$

Proof. Again, too lazy to read it. □

1.2.3 Central Character

Note 1.25. Hypothesis We will assume that G/K is countable for all compact open K whenever it makes life easy.

Lemma 1.26 (Schur)**.** *If (π, V) is smooth and irreducible representation of G, then* $\operatorname{End}_G(V) = \mathbb{C}$.

Proof. If $\phi \in \operatorname{End}_G(V)$, $\phi \neq 0$, then the image and kernel of ϕ are both G-subspaces of V, and thus so ϕ is bijective and invertible. So, $\operatorname{End}_G(V)$ is a complex division algebra. Now, $\dim_{\mathbb{C}}(V)$ is countable, and thus $\operatorname{End}_G(V)$ has countable dimension over $\mathbb{C}$. However, $\phi \notin \mathbb{C}$, then ϕ is transcendental over $\mathbb{C}$, and thus $\mathbb{C}(\phi) \subset \operatorname{End}_G(V)$ is uncountable. Thus, there is no such ϕ, and $\operatorname{End}_G(V) = \mathbb{C}$. □

Corollary 1.27. *Let (π, V) be an irreducible smooth representation of G. The center Z of G acts via a character* $\omega_\pi : Z \to \mathbb{C}^\times$.

Proof. We have a map $Z \to G \to \mathrm{Aut}_{\mathbb{C}}(V)$, and since Z is in the center, the image actually lies in $\mathrm{End}_G(V) = \mathbb{C}$. Thus, there is a homomorphism $\omega_\pi : Z \to \mathbb{C}^\times$ such that $\pi(z)v = \omega_\pi(z)v$ for $z \in Z$, $v \in V$. If K is a compact open of G such that $V^K \neq 0$, the ω_π is trivial on $K \cap Z$, as V^K has trivial K-action. Thus, ω_π is a quasi-character. □

Corollary 1.28. *If G is abelian, then any irreducible smooth representation of G is one-dimensional.*

Definition 1.29. The character $\omega_\pi : Z \to \mathbb{C}^\times$ is called the *central character* of π.

1.2.4 Semisimplicity

Proposition 1.30. *Let G be a locally profinite group and H be an open subgroup of G of finite index.*

1) *If (π, V) is a smooth representation of G, then V is semisimple if and only if it is H-semisimple.*
2) *Let (σ, W) be a semisimple smooth representation of H. Then, $\mathrm{Ind}_H^G \sigma$ is G-semisimple.*

Proof. Essentially uses finiteness of the index of H in G. □

1.2.5 Smooth/Contryagin Duality

Let (π, V) be a smooth representation of G. Let $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. Let

$$V^* \times V \to \mathbb{C}$$

$$(v^*, v) \mapsto \langle v^*, v \rangle$$

Then, V^* has an induced representation

$$\langle \pi^*(g)v^*, v \rangle = \langle v^*, \pi(g^{-1})v \rangle \qquad g \in G, v^* \in V^*, v \in V$$

If V^* were smooth, it would be a union of $(V^*)^K$. It is not in general, so we let

$$\check{V} = (V^*)^\infty = \bigcup_K (V^*)^K.$$

Then, $\check{V}$ is G-stable and is smooth.

Definition 1.31. Let $\check{pi} : G \to \mathrm{Aut}_{\mathbb{C}}(\check{V})$ be the representation on the subspace $\check{V} \subset V^*$. We call $(\check{\pi}, \check{V})$ to be the *contragradient* or *smooht dual* of (π, V). We have a pairing

$$\langle \cdot, \cdot \rangle : \check{V} \times V \to \mathbb{C}$$

and

$$\langle \check{\pi}(g)\check{v}, v \rangle = \langle \check{v}, \pi(g^{-1})v \rangle \qquad g \in G, \check{v} \in \check{V}, v \in V$$

Proposition 1.32. $\check{V}^K \cong (V^K)*$

Proof. We know that V^K is the unique K-complement of $V(K) = \{v - \pi(k)v | v \in V, k \in K\}$. If $\check{v} \in \check{V}$ is fixed under K, we must have

$$\langle \check{v}, V(K) \rangle = 0$$

by definition of $V(K)$. Indeed,

$$\langle \check{v}, v \rangle = \langle \check{\pi}\check{v}, v \rangle = \langle \check{v}, \pi(k^{-1})v \rangle$$

Then, $\check{v} \in \check{V}^K$ is determined by its action on V^K. One can extend any linear functional on V^K, which is a function of $V = V^K \oplus V(K)$ to an element of $\check{V}^K$ by declaring it to be trivial on $V(K)$. □

Corollary 1.33. *Let (π, V) be a smooth representation of G, $v \in V$, $v \neq 0$. There exists a $\check{v} \in \check{V}$ such that $\langle \check{v}, v \rangle \neq 0$.*

Proof. Let $v \in V$. There is some K such that $v \in V^K$. Then, by previous result, $(\check{V})^K = (V^K)*$ and we can choose something from here that pairs to something non-zero with v. □

Let (π, V) be a smooth representation of G. There is a canonical map

$$\delta : V \to \check{\check{V}}$$

given by

$$\langle \delta(v), \check{v} \rangle_{\check{V}} = \langle \check{v}, v \rangle_V \quad v \in V, \check{V} \in V$$

which is injective because of the last corollary.

Proposition 1.34. *The map $\delta : V \to \check{\check{V}}$ is an isomorphism if and only if (π, V) is admissible.*

Proof. The maps $\delta^K : V^K \to \check{\check{V}}^K$ for each compact open group K of G are surjective if and only if δ is. Now, δ^K is the usual double dual map

$$V^K \to (V^K)^{**}$$

which is surjective if and only if $\dim_{\mathbb{C}} V^K$ is finite. □

1.3 Integration Theory

Let $C_c^\infty(G)$ be the space of functions that are locally constant and have compact support. So, if $f \in C_c^\infty(G)$, there must exist K_1, K_2 (by local constancy) compact open subgroups (by support condition) of G such that f is constant on $K_1 g$ and gK_2. Thus, we see that if $K = K_1 \cap K_2$, then f is a finite linear combination of characteristic functions of the double cosets KgK.

Definition 1.35. Let

$$\lambda : G \times C_c^\infty(G) \to C_c^\infty(G) \qquad (g, f) \mapsto f(g^{-1}x)$$

$$\rho : G \times C_c^\infty(G) \to C_c^\infty(G) \qquad (x, f) \mapsto f(xg)$$

be the *left* and *right translation* actions respectively. These are smooth representations of G.

1.3.1 Haar Integrals

Definition 1.36. A *right Haar integral* on G is a non-zero linear functional

$$I : C_c^\infty(G) \to \mathbb{C}$$

such that

1) $I(\rho_g f) = I(f)$ for all $g \in G$, $f \in C_c^\infty(G)$
2) $I(f) \geq 0$ for any $f \in C_c^\infty(g)$, $f \geq 0$.

Similarly, one can define a *left Haar integral*.

Proposition 1.37. *There exists a right Haar integral* $I : C_c^\infty(G) \to C$. *Any other right Haar integral is a a multiple of* I *by some constant* $c > 0$. *Thus,*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(C_c^\infty(G), \mathbb{C}) = 1.$$

where we consider $C_c^\infty(G)$ *as a* G*-rep via right translation.*

Remark 1.38. $\mathrm{Hom}_G(C_\infty^c(G), \mathbb{C})$ is not the set of all Haar integrals.

Proof. Key idea is to see that ${}^K C_c^\infty(G)$ considered as a G-rep via Λ is a G-rep via ρ. Then, use compact induction. □

Corollary 1.39. *For* $f \in C_c^\infty(G)$, *define* $\check{f} \in C_c^\infty(G)$ *by* $\check{f}(g) = f(g^{-1})$ *for* $g \in G$. *Then, the functional*

$$I' : C_c^\infty(G) \to \mathbb{C}$$

$$I'(f) = I(\check{f})$$

is a left Haar integral. Any other left Haar integral is of the form cI', *with* $c > 0$.

1.3.2 Haar Measures

Let I be a left Haar integral and S a compact open subgroup. We define

$$\mu_G(S) = I(\mathbb{1}_S).$$

Then, $\mu_G(S) > 0$ and $\mu_G(gS) = \mu_G(S)$ for $g \in G$.

Definition 1.40. We refer μ_G as the *left Haar measure*. We denote the relation between Haar integrals and Haar measures by

$$I(f) = \int_G f(g) d\mu_G(g)$$

for $f \in C_c^\infty(G)$. We say that G is *unimodular* if any left Haar integral on G is a right Haar integral.

Using similar techniques, we can define Haar integrals on $C_c^\infty(G, V) = C_c^\infty(G) \otimes_{\mathbb{C}} V$ i.e if $\phi \in C_c^\infty(G; V)$, we can write

$$I_V(\phi) = \int_G \phi(g) d\mu_G(g).$$

Let μ_G be a left Haar measure on G. For $g \in G$, consider

$$C_c^\infty(G) \to \mathbb{C}$$
$$f \mapsto \int_G f(xg) d\mu_G(x).$$

This is also a left Haar integral, and thus there is a unique $\delta_G(g) \in \mathcal{R}_+^\times$ such that

$$\delta_G(g) \int_G f(xg) d\mu_G(x) = \int_G f(x) d\mu_G(x)$$

for all $f \in C_c^\infty(G)$. The function $\delta_G : G \to \mathbb{R}_+^\times$ is a homomorphism. If we take f to be the charactersitic function of a compact open, then δ_G is trivial on K. Thus, δ_G is a quasi-character. It is trivial if and only if δ_G is unimodular, since

$$f \mapsto \int_G \delta_G(x)^{-1} f(x) d\mu_G(x)$$

for $f \in C_c^\infty$ is a right Haar integral.

Remark 1.41. We can make the mnemonic

$$d\mu_G(xg) = \delta_G(g) d\mu_G(x)$$

Definition 1.42. We call δ_G the *module* of G.

1.3.3 Duality

1.4 Hecke Algebra

Let G be a locally profinite group. Smooth representations (π, V) of G are algebras not over $\mathbb{C}[G]$, but what is called the *Hecke algebra*.

Hypothesis G is unimodular.

Fix a Haar measure μ on G. For $f_1 \in f_2 \in C_c^\infty(G)$, we define

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) d\mu(x).$$

Indeed, the function $(x, g) \mapsto f_1(x) f_2(x^{-1}g) \in C_c^\infty(G \times G)$ which implies that $f_1 * f_2 \in C_c^\infty(G)$. We can check that

$$f_1 * (f_2 * f_3) = (f_1 * f_2) f_3.$$

Then, the pair

$$\mathcal{H}(G) = (C_c^\infty(G), *)$$

is an associative $\mathbb{C}$-algebra, which we call the *Hecke* algebra of G. Notice that $\mathcal{H}(G)$ has no unit element unless G is discrete, but it has a lot of idempotents.

Remark 1.43. If we replace μ with $\nu = c\mu$, then the corresponding Hecke algebras $\mathcal{H}_\mu(G)$ and $\mathcal{H}_{nu}(G)$ are isomorphic via $f \mapsto c^{-1}f$ for $f \in \mathcal{H}_\mu(G)$.

Example 1.44. Let G be discrete. The counting measure is then a Haar measure on G. Let's us write

$$\int_G f(g)d\mu(g) = \sum_{g \in G} f(g)$$

Thus, the map

$$\begin{aligned} \mathcal{H}(G) &\to \mathbb{C}[G] \\ f &\mapsto \sum_{g \in G} f(g)g \end{aligned}$$

is an isomorphism.

1.4.1 Idempotents

For each compact open K of G, define $e_K \in \mathcal{H}G$ to be $\mu(K))^{-1}\mathbb{1}_K$.

Proposition 1.45. *The following are true.*

1) $e_K * e_K = e_K$
2) $f \in \mathcal{H}G$*, satisfies* $e_K * f = f \iff f(kg) = f(g)$ *for all* $k \in K$*,* $g \in G$
3) *The space* $e_K * \mathcal{H}G$ *is a sub-algebra of* $\mathcal{H}G$ *with unit element* e_K*.*

Proof. We have

$$\begin{aligned} e_K * e_K &= \int_G e_K(x)e_K(x^{-1}g)d\mu(x) \\ &= \int_K e_K(x)e_K(x^{-1}g)d\mu(x) + \int_{G \setminus K} e_K(x)e_K(x^{-1}g)d\mu(x) \\ &= \mu_K \cdot \mu_K^{-2} \cdot \mathbb{1}_K \\ &= e_K \end{aligned}$$

Similar concrete computations can show that $e_K * f$ is left K-invariant. □

1.4.2 Smooth modules over Hecke Algebra

Let M be a left $\mathcal{H}G$ module, where we denote the multiplication by $*$ i.e. $f \in \mathcal{H}G$, $m \in M$, $(f, m) \mapsto f * m$.

Definition 1.46. We say that a module M is *smooth* if $\mathcal{H}G * M = M$.

Since

$$\mathcal{H}G = \bigcup_K (e_K * \mathcal{H}(G) * e_K)$$

M is smooth if and only if for every $m \in M$, there is a compact open subgroup K of G such that $e_K * m = m$. We define $\mathrm{Hom}_{\mathcal{H}G}(M_1, M_2)$ for smooth modules M_1, M_2 in the obvious way. Then, we get a category $\mathcal{H}G-\mathrm{mod}$ of smooth modules.

Definition 1.47. Let (π, V) be a smooth G-rep. For $f \in \mathcal{H}G$, $v \in V$, let

$$\pi(f)v = \int_G f(g)\pi(g)\, v\, d\mu(g) \in C_c^\infty(G; V).$$

Proposition 1.48. *Let (π, V) be a smooth representation of G. Then, under*

$$(f, v) \mapsto \pi(f)v \qquad f \in \mathcal{H}(G), v \in V$$

gives V the structure of a smooth $\mathcal{H}G$-module. Conversely, given a smooth $\mathcal{H}G$-module M, there is a unique G-homomorphism $\pi : G \to \mathrm{Aut}_{\mathbb{C}}(M)$ given by

$$\pi(f)m = f * m$$

such that (π, M) is a smooth representation of G and this construction behaves well with respect to homomorphisms. So, in particular the category $\mathrm{Rep}(G)$ is then equivalent to $\mathcal{H}(G) - mod$ category.

Proof. We need to check that $\pi(f_1 * f_2) = \pi(f_1)\pi(f_2)$ for $f_1, f_2 \in \mathcal{H}(G)$. This can be done formally. The fact that V is smooth as a $\mathcal{H}(G)$-module can be deduced from the fact that if K is compact open in G that fixed v and f (under right translation), then

$$\pi(f)v = \mu(K) \sum_{g \in G/K} f(g)\pi(g)v$$

which is a finite sum since f has compact support. From this, we see that

$$\pi(e_K)v = v$$

and hence, V is smooth as a $\mathcal{H}G$-module. □

1.4.3 Subalgebras

Let K be a compact open subgroup. Then, $e_K * \mathcal{H}G * e_K$ is the space of K bi-invariant locally constant compactly supported functions on G with $*$ as involution. It has an identity. We denote this sub-algebra by $\mathcal{H}(G, K)$.

Lemma 1.49. *Let (π, V) be a smooth rep. The operator $\pi(e_K)$ is the K-projection $V \to V_K$ with kernel $V(K)$. The space V^K is a $\mathcal{H}(G, K)$-module on which e_K acts as identity.*

Proof. For $v \in V$, $k \in K$, we have

$$\pi(k)\pi(e_K)v = \pi(e_K)\pi(k)v = \pi(e_K v)$$

by writing down all the integrals. Thus $\pi(e_K)$ is a K-map $V \to V^K$. As $\pi(e_K)v = v$ for all $v \in V^K$, the image of $\pi(e_K) : V \to V^K$ is all of V^K. Now, $\pi(e_K)$ is an idempotent, and $\pi(e_K)$ annihilates the unique K -complement $V(K)$ of V^K. □

Proposition 1.50. *1) Let (π, V) be a smooth irreducible representation of G. Then, V^K is either 0 or a simple $\mathcal{H}(G, K)$ module.*

2) The process $(\pi, V) \mapsto V^K$ induces a bijection between the following sets of objects

$$\left\{equivalence\ classes\ of\ irre\ smooth\ reps (\pi, V) of G with V^K \neq 0\right\}$$

and

$$\{isomorhpism\ classes\ of\ simple\ \mathcal{H}(G, K)\text{-}modules\}$$

Corollary 1.51. *Let (π, V) be a smooth representation of V such that $V \neq 0$. Then, (π, V) is irreducible if and only if for just one hence any compact open subgroup K of G, the space V^K is either zero or a simple $\mathcal{H}(G, K)$-module.*

Proof. $\implies$ is obvious. Suppose $(\pi, V)$is not irreducible, and let $U \subsetneq V$ be a G-stable subspace. Set $W = V/U$. There is a compact open subgroup K of G such that both spaces W^K and U^K are non-zero. The sequence

$$0 \to U^K \to V^K \to W^K \to 0$$

is exact and is an exact sequence of $\mathcal{H}(G, K)$-modules. Thus, V^K is non-zero and non-simple over $\mathcal{H}(G, K)$. □