Categorical Local Langlands:

[up to renormalization].

$$G(K)\text{-mod} \simeq \mathrm{ShvCat}(\mathrm{LocSys}_{\check{G}}(\mathring{D})).$$

De-categorify ~~[illegible]~~

(nice) irred representations ($/\bar{\mathbb{Q}}_\ell$ or $\mathbb{C}$, $\ell \neq p$). (For this talk, $\mathbb{C}$).
of ~~[illegible]~~ group $G(F)$, F local field.
(really, L-packets of them)

$\longleftrightarrow$ bijection satisfying long list of compatibilities.

Maps of Weil-Deligne grp W_F' to ${}^LG = \check{G} \rtimes \mathrm{Gal}(\bar{F}/F)$
(really, modifications thereof).

(We seem to have decategorified twice: $\exists$ cat. description (Fargues' Conjecture, etc.) but let's not go there). (c.f. Vogan).

restrict to GL_n to get rid of the complications.

$\left\{ \begin{array}{l} \text{irred. smooth admissible} \\ \text{rep of } GL_n(F) \end{array} \right\}$ $\longleftrightarrow$ bij satisfying compatibilities. $\left\{ \begin{array}{l} \text{smooth Frobenius-semisimple} \\ \text{rep of } W_F \text{ (the Weil group)} \end{array} \right\}$.

$W_F := \mathrm{val}^{-1}(\mathbb{Z})$ in

$$0 \to I_F \to \mathrm{Gal}(\bar{F}/F) \xrightarrow{\mathrm{val}} \mathrm{Gal}(\bar{k}/k) = \varprojlim \mathbb{Z}/n = \hat{\mathbb{Z}} \to 0$$

This is a theorem for all F.

Archimedean: Langlands.

non-Archimedean:
- $\mathbb{F}_q((t))$: Laumon, Rapoport and Stuhler.
- $[E:\mathbb{Q}_p]<\infty$: Harris & Taylor, Henniart / Scholze ('10)

we'll focus on those.

mostly these

For majority of this semester we're going to stay with $n=2$. (I'll remind you in a see what $n=1$ is.). This is Jacquet & Langlands (1970). ~~[illegible]~~.
(For this talk $p \neq 2$.)
Whereas general proof for GL_n uses global techniques (Shimura varieties, Drinfel'd / LN Týkoas etc.) this case seems to be sufficiently "local in method".

The way we prove this is to identify the two sides explicitly and match them up. The RHS ~~is relatively easy to describe. Namely, they~~ correspond to 2-d Weil-Deligne reps of W_F, which are pairs (V, N):

V is a 2d vector space w/ W_F action, s.t. I_F acts via a finite quotient,
$\widetilde{\mathrm{Frob}}$ (some lift of Frob) acts diagonalizably,
and $N \in \mathrm{End}(V)$. $gNg^{-1} = |k|^{\mathrm{val}(g)} N$.

(More can be said e.g. l-adic monodromy. Not now.)

And these reps are easy to classify: it's either

- V = a direct sum of two characters of W_F, and $N=0$.
- V irred. $N=0$.
- $V = W \oplus W(1)$. $NW(1)=0$, $NW \simeq W(1)$,
  $W(1) := W \otimes (g \mapsto q^{\mathrm{val}\, g})$.

Classification of the LHS is our first goal in mind. We need to define it:

A rep. (often inf-dim!) of $GL_n F$ is

swap $\updownarrow$
- admissible if $\forall$ open subgp U, V^U is f.d. (dualizable).
- smooth if $\mathrm{Stab}(v)$ is open for all v. (not too big)

Case of $n=1$

In case of $n=1$, $GL_1 F = F^\times$, so we have a character $F^\times \to \mathbb{C}^\times$ by Schur's Lemma

~~RHS on the other hand is~~

the kernel being open means it contains $1+\varpi^n \mathcal{O}$ for some $n>0$.

What about ~~R~~ W_F - reps? Well, it's a continuous hom. $W_F \to \mathbb{C}^\times$.

And recall we have the Artin map $\theta_F : W_F \to F^\times$.

$$\begin{array}{ccc} W_F & \longrightarrow & F^\times \\ \downarrow & & \wr\!\mid \\ & \mathrm{Gal}(F^{ab}/F) & \end{array}$$

s.t. $\theta_F(I_F) = \mathcal{O}^\times$ and $(\theta_F(x)$ is a uniformizer$) \Longleftrightarrow (x$ lifts Frob$)$.

It follows directly that θ_F does the job. (Though I didn't specify the compatibilities)

Back to main story though. 2-d admissible smooth reps of $GL_2(F)$. How do they look like?

Turns out also a trichotomy:

- principal series rep $V(\chi_1, \chi_2)$ coming from two characters. (or the 1-d summand of $V(\chi_1,\chi_2) = \square \oplus st\otimes\chi$ in degenerate case)
- ~~Twisted Steinberg~~ Supercuspidal reps (see below). ~~(we construct these otherwise)~~
- Twisted Steinberg reps $st \otimes \chi$.

and they will match (in this order) w/ the trichotomy given above.

This is the baby ~~e~~ case of the Bernstein-Zelevinsky ~~classification~~ ~~(for GL_n) / Jacquet / general~~ classification, which says in spirit that

Everything can be built out of <u>supercuspidal</u> reps via <u>parabolic induction</u>. <u>uniquely</u>!

Note: there isn't such thing in Archimedean cases.

Let's be precise. ~~(Here G is general)~~

Fix P a parabolic, and $P = MN$ the Levi decomposition.

For $\rho: P \to V$ a smooth rep, define $\rho_N := V / \{nv - v \mid n \in N\}$.

This $(\rho \mapsto \rho_N)$ defines the Jacquet functor $J(\rho)$.
(w/ a twist)

and we'll check that it has many nice properties. e.g. it being an exact additive functor from smooth (adm) to smooth (adm.), and has Frobenius reciprocity. We can define super-cuspidal reps as those whose ~~Jacquet~~ image under Jacquet functors are all 0.

There are other ways to characterize this, e.g. by the compact supportedness of one (and thus every) matrix coefficient of the representation, or that it doesn't appear as a subquotient of sth parabolically induced from a proper subgroup.

(matrix coeff. $f_{v,v^\vee}$:
$(\pi, V) \xrightarrow[\text{contragred. dual}]{\sim} (\pi^\vee, V^\vee)$ $v \in \pi$, $v' \in \pi$, $f_{v,v'}(g) = \langle v', \pi(g) v$

<u>Thm</u> Every irred smooth ($\Leftrightarrow$ admissible) rep of $G(F)$ appears as a subrep in the parabolic induction of some supercuspidal rep. (For this talk, let's cheat and define parabolic induction as the right adjoint of Jacquet.) (up to some equivalence, it shows up uniquely.)

But we can do much better than this.

We can define the <u>discrete series</u> of G (aka essentially square integrable reps), which are a ~~slightly~~ larger class of irred reps characterized by their matrix coefficients being square-integrable.

<u>Thm.</u> Every discrete series rep is equivalent (in a unique way) to the unique irreducible quotient of parab. induction from $GL_a F \times \cdots \times GL_a F$ for $n = a + \cdots + a$, some $\Delta = \sigma \otimes \sigma(1) \otimes \cdots \otimes \sigma(\frac{n}{a} - 1)$, for σ irred. supercuspidal. Denote it by $Q(\Delta)$

And finally,

Thm. under some constraint on $\Delta_1 \cdots \Delta_k$,

Every irred. rep is uniquely given as the unique irred quotient of some $\mathrm{Ind}_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_k))$.

Thus in general, the classification of irred. rep. reduces to that of the supercuspidal ones, ~~\[illegible\]~~ and to construct Langlands it suffices to cook up Weil-Deligne reps from the supercuspidal ones. In general this is a lot of work — and when the global geometry comes in — but for GL_2 we can do this "by hand" (we should go through the Weil rep. construction though.).

Note: we could tackle $p=2$ but that's a lot of extra work.

Functional Equation.

I have yet to ~~\[illegible\]~~ explain the condition that the bijection needs to satisfy.

The main players are the L-functions $L(\pi, s)$ and the ~~\[illegible\]~~ local constant $\varepsilon(\pi, s, \psi)$.

s a complex variable.
ψ a nontrivial character of F.

π either an automorphic form or a Galois rep.

Their definition is going to take ~~\[illegible\]~~ a lot of time (esp. the existence of ε) and we'll need to prove functional equations about them (to be able to bootstrap from the supercuspidals).

Let me just copy some words from the book: in GL_2, we have the following phenomena: (what's the general story?).

1) as χ ranges over characters of F, L and ε of $\chi \otimes \pi$ actually determines π uniquely;

2) When π is not ~~supercuspidal~~ supercuspidal, L is constantly 1, and π is uniquely determined by ε;

3) otherwise, π is ——— by L instead.

Local vs Global (aka notable objects)

The discussion would be largely incomplete w/o mentioning how the story interacts w/ the global one. For now, let us just give some keywords on possible directions.

- Generic representation i.e. those admitting a (unique) Whittaker Model. (Thm)
  - in particular, supercuspidals are.
  - discrete series are as well. ← only for GL
  - local components of cuspidal global obj are generic. (conj: tempered).
- tempered rep
  - slightly smaller space (ess. sq. integrable vs. sq. int.) than generic

- unramified (almost all local factors of any global obj.).
  - Those admitting a $G(O)$-invariant.
  - uniquely identified by the Hecke character

  $$\Big(\underbrace{\mathcal{H}_K}_{\text{commutative}} \curvearrowright G(O)\text{-invariant of } V \Big).$$

  - All come from $Q(\chi_1, \dots \chi_n)$ satisfying some condition.

~~Ramanujan: local comp of global cuspidal are unramified.~~

- Iwahori (having I invariant).
  - Same as usual def. I = inverse image of Borel under $GL_n(O) \to GL_n(k)$.

<u>Thm</u> (Borel, Casselman).

$B = TU$.

(π, V) admissible.

then $V \twoheadrightarrow V_U$

induces $V^I \cong (V_U)^{I^0}$. (For us: $C^{M(K)T(O)} = C^I$,

<u>Thm</u> $\chi_1 \dots \chi_n$ unramified characters.

irred. quot π of $\pi(\chi_1, \dots \chi_n)$ principal series (Parabolic induced from Borel)

$\Updownarrow$

Iwahori