Feb 9 (Yifei) Whit $\longrightarrow$ Opers.

Consider $\mathrm{Whit}(\mathrm{Gr}_G)$. (ev. thing over a point).

$\simeq \mathrm{Dmod}^*(\mathrm{Gr}_G)^{N(K), \chi}$.

today: $\downarrow$ (Fibre at ∞).

$\mathrm{QCoh}(\mathrm{Op}_{PG}^{\mathrm{unr}})$.

(we forget about E in this talk)

$$\mathrm{Op}_{P_G}(X) = \{(\underbrace{P_G, \nabla}_{\text{conn on } P_G}, \underbrace{P_B}_{\text{reduction to } B}, (P_B)_T \xrightarrow{\sim} \omega_X^\rho\}.$$

Fix $\omega_X^{1/2}$ $\quad 2\rho : \mathbb{G}_m \to T$.

Satisfying $\quad \omega_X^\rho : (\omega_X^{1/2})^{2\rho}$: T-bundle induced.

Oper condition:

(∇ doesn't respect $P_G \rightsquigarrow P_B$ reduction.)

define $\nabla / P_B \in (\mathfrak{g}/\mathfrak{b})_{P_B} \otimes \omega_X$

$$0 \to \mathfrak{g}_{P_G} \longrightarrow \mathrm{At}(P_G) \longrightarrow T_X \to 0$$

G-inv. vector field
on total space
relative tangent complex of $X \to BG$

$$
\begin{tikzcd}
0 \arrow[r] & \mathfrak{b}_{P_B} \arrow[r] \arrow[d] & At(P_B) \arrow[r] \arrow[d] & T_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{g}_{P_B} \arrow[r] \arrow[d] & At(P_G) \arrow[r] \arrow[d] & T_X \arrow[r] \arrow[l, bend right, "\nabla"'] \arrow[dl, bend left] & 0 \\
 & (\mathfrak{g}/\mathfrak{b})_{P_B} \arrow[r, equal] & (\mathfrak{g}/\mathfrak{b})_{P_B} & &
\end{tikzcd}
$$

$(= \mathfrak{g}_{P_G})$

then: $T_X \xrightarrow{\nabla} At(P_G) \longrightarrow (\mathfrak{g}/\mathfrak{b})_{P_B}$.

Oper element:

$$
\begin{array}{ccc}
\chi \in (\mathfrak{b}_{-1}/\mathfrak{b})_{P_B} \otimes \omega_X & \xrightarrow{\sim} & (\mathfrak{b}_{-1}/\mathfrak{b})_{\omega_X^{e}} \otimes \omega_X \\
\uparrow & & \cong \\
(1,1,\dots,1) & \text{(Factor thru } T\text{)} & \bigoplus_{i\in\Delta} T_X \otimes \omega_X \\
\downarrow & & \cong \\
(\mathfrak{g}/\mathfrak{b})_{P_B} \otimes \omega_X & & \bigoplus_{i\in\Delta} \mathcal{O}_X
\end{array}
$$

Remark: often defined as

$$\chi \longrightarrow (\mathfrak{b}_{-i})/\mathfrak{b} \text{ iso for each } i \in \Delta.$$

Remark: G of adj type then last iso is redundant.

(b/c simple roots span root lattice in adjoint type)

Variants:

1) $X \longrightarrow$ formal disc/punctured.

2) $\mathrm{Op}^{unr} := \{$ similar data, where P_G, ∇ defined on formal disc, P_B, iso defined on punctured disc thus unramified as G-local sys. $\}$.

$$\nabla/_{P_B} \in (\mathfrak{g}/\mathfrak{b}) \otimes \overset{\circ}{\omega}_x$$
(punctured)

Remark:

recall

$$\begin{tikzcd}
\mathrm{KL}_G^k \arrow[r, "\mathrm{FLE}"] \arrow[d] & \mathrm{Whit}^{\check{E}}_{\check{G}} \arrow[d] \\
\mathrm{Dmod}^k(\mathrm{Bun}_G) \arrow[r, "\mathrm{LG}"] & \mathrm{Dmod}^{-\check{k}}(\mathrm{Bun}_{\check{G}})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{KL}_{crit} \arrow[r, "\mathrm{FLE}", "\sim"'] \arrow[d] & \mathrm{QCoh}(\mathrm{Op}^{unr}_{\check{G}}) \arrow[d] \\
\mathrm{Dmod}_{crit}(\mathrm{Bun}_G) \arrow[r] & \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})
\end{tikzcd}$$

Thm: Frenkel–Gaitsgory (spherical)

eqv of abelian cats

In particular:

$$\mathrm{Vac} \rightsquigarrow \mathcal{O}_{\mathrm{Op}_{\check{G}}}(D_x)$$

One can recover FF from this

Further remark:

$$\begin{matrix} \mathrm{Whit}^{\mathrm{ext}} \\ \uparrow \\ \mathrm{Dmod} \end{matrix} \quad \text{Poincaré (coeff.)} \quad \text{coefficient}$$

at ∞-level:

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \longrightarrow \mathrm{QCoh}(\mathrm{Op}_{\check{G}}^{\text{possibly deg}})$$

Conj:

Ran version of these are fully faithful.

QCoh $\hookrightarrow$ QCoh part is

now a thm of Arinkin.

(Main missing ingredient is

~~the generalization of this~~)

$\mathrm{Dmod}^{\kappa}(G \times G)^{LN, \chi}$

$\kappa \in \mathrm{Par}_G = \{ G\text{-inv Lagrangian subspaces } \mathfrak{g}^{\kappa} \text{ in } \mathfrak{g} \oplus \mathfrak{g}^* \}.$

$\{\mathfrak{g}^{\infty} = \mathfrak{g}^*\} \in \quad \cup \quad \mathrm{Par}^{\circ}_G = \left\{ \begin{array}{c} \mathfrak{g}^{\kappa} \\ \downarrow \text{projection iso} \\ \mathfrak{g} \end{array} \right\}$ then it's a graph.

Motto: replace all $\mathfrak{g}$ by $\mathfrak{g}^{\kappa}$.

$\mathfrak{g}^{\kappa}$ comes w/ symmetric form

so can define $\curvearrowleft LG$

$$0 \to k\mathbb{1} \to \hat{\mathfrak{g}}^{\kappa} \to L\mathfrak{g}^{\kappa} \to 0$$

Example.

a) $\hat{\mathfrak{g}}\text{-mod} := \hat{\mathfrak{g}}^{\kappa}\text{-modules on which } \mathbb{1} = 1.$

Prop. $\hat{\mathfrak{g}}^{\infty}\text{-mod} \simeq \mathrm{QCoh}(\mathrm{Conn}(\overset{\circ}{D}_x))/G$

Proof: $\hat{\mathfrak{g}}^{\infty}\text{-mod} \xrightarrow{\sim} U(\hat{\mathfrak{g}}^{\infty}) / (\mathbb{1} - 1)$

$\simeq \mathrm{Sym}(k\mathbb{1} \oplus \mathfrak{g}^*((x))) / (1 - \mathbb{1}).$

$\cong$ alg of functions on

$$\begin{array}{c} \mathbb{A}^1 \times (\underbrace{\mathfrak{g} \otimes \omega_x^\bullet}_{\text{dual of } \mathfrak{g}^*(K_x)}) \\ \text{(take fiber of 1)} \Big\downarrow \\ \mathbb{A}^1 \end{array}$$

get $d + \mathfrak{g} \otimes \omega_x^\bullet$.

note: this means LG action is respected.

side: why is $\hat{g}^\kappa$ Lie algebra?

$$[\xi \oplus \varphi, \xi' \oplus \varphi'] = [\xi, \xi'] \oplus \underset{\xi}{\mathrm{coad}}(\varphi').$$

Lagrangian + G-inv $\rightsquigarrow$ it's actually anti-symmetric.

then $\mathrm{Vac}^\infty = \mathcal{O}_{\mathrm{Conn}}(D_x)$.

b) $\mathrm{Dmod}^\kappa(LG)$

Courant ext. of Lie algebroids

$$0 \to \mathcal{O}_{LG}\mathbb{1} \to \hat{g}^\kappa \otimes \mathcal{O}_{LG} \to Lg^\kappa \otimes \mathcal{O}_{LG} \to 0$$

$$\mathrm{Dmod}^\kappa(LG) := \hat{g}^\kappa \otimes \mathcal{O}_G\text{-mod}/\mathbb{1} = \mathrm{id}.$$

then $\mathrm{Dmod}^\infty(LG) = \mathrm{QCoh}(LG \times \mathrm{Conn}_G(\mathring{D}_x))$.

same as deformation of Dmod to QCoh on cotangent bundle.

c) $\mathrm{Dmod}^\kappa(\mathrm{Gr}_G) := \mathrm{Dmod}^\kappa(LG)^{(L^+G)^\kappa}$.

Recall

$$1 \to \exp(L^+g) \to L^+G \to (L^+G)_{dR} \to 1$$

replace w/

$$\exp(L^+g)^\kappa \to L^+G \to (L^+G)^\kappa \to 1$$

$$\searrow \exp(L^+g) \nearrow$$

note $\kappa \neq \infty$, then $L^+G^\kappa = L^+G_{dR}$.

what is $(L^+G)^\infty$?

$$\exp(L^+g^*) \xrightarrow{1} L^+G \to L^+G \ltimes B\exp(L^+g^*)$$

Claim: $\mathrm{Dmod}^\infty(\mathrm{Gr}_G) = \mathrm{DMod}^\infty(LG)^{B\exp(L^+g^*) \rtimes L^+G}$

$$= \left(\mathrm{Dmod}^\infty(LG)^{B\exp(L^+g^*)}\right)^{L^+G}$$

Lem

$$\begin{array}{ccc} \exp(\mathfrak{g}) & & \text{Lie algebroid } L \\ \downarrow & \curvearrowright & \downarrow \\ H & & Y \\ \downarrow & & \\ H^b := H/\exp(\mathfrak{g}). & & \end{array} \quad \rightsquigarrow \quad \begin{array}{c} Y \\ \downarrow \\ Y^b \end{array} \text{ formal moduli.}$$

$\mathrm{IndCoh}(Y^b) \simeq \mathcal{L}\text{-mod}.$

lemma 1) extension of action $\Longleftrightarrow \left\{ \mathfrak{g} \otimes \mathcal{O}_Y \longrightarrow L \;\; H\text{-eqv. satisfying some conditions} \right\}.$

2) Specialize to $H = pt$, $\mathfrak{g}$ abelian.
$\mathcal{L}$ abelian.

then

$$\begin{array}{ccc} & \mathrm{IndCoh}(Y^b)^{B\exp(\mathfrak{g})} & \\ \simeq \swarrow & & \searrow \simeq \\ \mathrm{QCoh}(\underbrace{\mathbb{V}(L)}_{\mathrm{Spec\,Sym}\,L})^{B\exp(\mathfrak{g})} & & \mathrm{QCoh}(\mathbb{V}(\mathrm{cofib}(\mathfrak{g} \otimes \mathcal{O}_Y \to L))) \end{array}$$

(continue claim)

$\xrightarrow{\sim} \left(\mathrm{QCoh}(LG \times \mathrm{Conn}_G(D^\circ))^{B\exp(L^+\mathfrak{g}^*)}\right)^{L^+G}$

$\xrightarrow{\sim} \left(\mathrm{QCoh}(LG \times \mathrm{Conn}_G D)\right)^{L^+G}.$

$L^+\mathfrak{g}^* \otimes \mathcal{O}_{LG} \to L\mathfrak{g}^* \otimes \mathcal{O}_{LG}$

$\mathrm{cofib} = L\mathfrak{g}^* / L^+\mathfrak{g}^*$

dualizes to $\mathfrak{g} \otimes \omega$.

$\xrightarrow{\sim} QCoh\left(\left\{\begin{array}{c}\text{classifying prestack for} \\ (P_G, P_G|_{\mathring{D}_x} \xrightarrow{\sim} P_G^\circ) \\ \text{w/ } \nabla \text{ on } P_G\end{array}\right\}\right)$

Let's call this Gr_G^{conn}.

Ex) Define KL_G^κ and show $KL_G^\infty \xrightarrow{\sim} Rep\, G$.

Now let's do Whittaker.

$Dmod^\kappa(Gr_G)^{\text{"}LN,\chi\text{"}}$ ... not quite.

$$\begin{tikzcd} \exp(L\mathfrak{g}^\kappa) \arrow[r] & LG \arrow[r] & (LG)^\kappa \\ \exp(L\mathfrak{n}^\kappa) \arrow[r] & LN \arrow[r] \arrow[u] & (LN)^\kappa \arrow[u] \end{tikzcd}$$

$\mathfrak{n}^\kappa \subset \mathfrak{n} \oplus \mathfrak{b}^\perp \leftarrow$ functionals on $\mathfrak{g}$ vanishing on $\mathfrak{b}$.

$\quad \shortparallel$

$\mathfrak{g}^\kappa \cap (\mathfrak{n} \oplus \mathfrak{b}^\perp)$.

Lemma $\exp(\mathfrak{a}) \longrightarrow H \longrightarrow H/\exp(\mathfrak{a})$

then $\left\{\begin{array}{c}H\text{-eqv} \\ \text{Lie alg} \\ \text{characters} \\ \text{of } \mathfrak{a}\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\mathbb{G}_m\text{-multiplicative} \\ \text{line bundles on} \\ H/\exp(\mathfrak{a}) \text{ w/} \\ \text{trivialization on } H\end{array}\right\}$

So suffices to have character $L\mathfrak{n}^\kappa \to k$.

cannot pick Chevalley generators on

$L\mathfrak{n}^\kappa \longrightarrow L\mathfrak{n}^\kappa/[\mathfrak{n}, \mathfrak{n}^\kappa]$.

No global section for $\mathbb{P}^1(-1)$.

so twist entire thing by ω_x^{ρ}.
then get canonical character.

$$L\mathfrak{n}_w^{+} \longrightarrow L\mathfrak{n}_w^{k}/[L\mathfrak{n}_w^{k}, L\mathfrak{n}_w^{k}]$$

$$\xrightarrow{\sim} \bigoplus_{i \in \Delta} \omega_x \xrightarrow{\Sigma \mathrm{Res}} k.$$

at ∞

$$L\mathfrak{b}_w^{+}/(L\mathfrak{b}_{(-1)}^{+})_w \cong L(\mathfrak{b}_{(-1)}/\mathfrak{b})_w^{*}.$$

so a character
gives $(\mathfrak{b}_{-1}/\mathfrak{b})_w \otimes \omega^{\bullet}$.

ex ~~the~~ Define $\mathrm{Dmod}^k(\mathrm{Gr}_G)^{\mathcal{L}N^k}$

and $k = \infty \Rightarrow$ $\mathrm{QCoh}(\mathrm{LocSys}(D_x) \times_{\mathrm{LocSys}(\overset{\circ}{D}_x)} \mathrm{LocSys}_B(\overset{\circ}{D}_x))$

$\simeq \tilde{\mathfrak{g}}/\mathfrak{g}$.