Nearby Cycles of Whittaker Sheaves. (Justin)

Outline.

I. nearby cycles.

II. example of $\mathbb{P}^1$.

III. main results.

IV. relevance to Langlands.

Notation: k alg closed char 0. (for char p use ℓ-adic sheaves.)
(not essential)

Y = scheme, stack, etc. $D(Y) :=$ (unbounded derived) cat of D-mod's on Y.

$\cup$

$D_{hol}(Y) :=$ bounded complexes w/ holonomic cohomologies.

Nearby Cycles.

$$\begin{tikzcd} Y_0 \arrow[r, hook, "i"] \arrow[d] & Y \arrow[d, "f"] & \overset{\circ}{Y} \arrow[l, hook', "j"'] \arrow[d] \\ \{0\} \arrow[r] & \mathbb{A}^1 & \mathbb{G}_m \arrow[l] \end{tikzcd}$$

$\rightsquigarrow i^! j_! : D_{hol}(\overset{\circ}{Y}) \to D_{hol}(Y_0)$.

(or $i^* j_*$, but not $i^! j_*$ since then vanish.)
by base change

(Thus) don't have this in function-theoretic setting.

<u>Remark:</u> $D_{hol}(Y)$ can be formally recovered from $i^! j_!$.

(lax limit of $i^! j_! : D_{hol}(\overset{\circ}{Y}) \to D_{hol}(Y_0)$).

M in $D(\mathring{Y}) \rightsquigarrow H^{\cdot}_{dR}(\mathring{Y}) \circlearrowright M$.

$H^{\cdot}_{dR}(\mathbb{G}_m)$. $\nearrow$ ($H^{\cdot}_{dR}$ = end (id) of monoidal structure $\overset{!}{\otimes}$).

$\rightsquigarrow H^{\cdot}_{dR}(\mathbb{G}_m) \circlearrowright i^! j_! M$.

$\{1\} \hookrightarrow \mathbb{G}_m \rightsquigarrow H^{\cdot}(\mathbb{G}_m) \circlearrowright k$ (augmentation)

(Unipotent) nearby cycles

functor.

$\Phi : D_{hol}(\mathring{Y}) \longrightarrow D_{hol}(Y_0)$.

$$\Phi(M) := k \underset{H^{\cdot}\mathbb{G}_m}{\otimes} i^! j_! M.$$

$$\text{“} = \lim_{t \to 0} M|^!_{Y_t} [1] \text{”}.$$

$\mathrm{End}_{H^{\cdot}(\mathbb{G}_m)}(k) \cong k[[t]] \circlearrowright \Phi$. is called the monodromy. (locally nilpotent).

<u>Lemma</u> This construction of Φ agrees w/ Beilinson's Construction.

In particular:

i) it is t-exact. ($i^! j_!$ is only left t-exact)

ii) it commutes w/ proper direct image and smooth inverse image.

iii) it commutes w/ Verdier duality. ($i^! j_!$ doesn't)

iv) if $\mathbb{G}_m$ acts on Y s.t. $f: Y \to \mathbb{A}^1$ is $\mathbb{G}_m$-equivariant, then.

M $\mathbb{G}_m$-equivariant $\Rightarrow$ $\Phi(M)$ is unipotently $\mathbb{G}_m$-monodromic.

w/ the monodromy equal to obstruction to $\mathbb{G}_m$-equivariance.

eg if $\mathcal{M}$ is a lisse D-mod on Y. then $\Phi(\mathcal{M}/\mathring{y}) = \mathcal{M}|^!_{y_0}[1]$.
(vector bundle)
w/ flat conn.

Example of $\mathbb{P}^1$.

For any $t \in k \rightsquigarrow e^{tx} \in D(\mathbb{A}^1)$.

$$\Gamma(\mathbb{A}^1, e^{tx}) = k\langle x, \tfrac{\partial}{\partial x}\rangle / (\tfrac{\partial}{\partial x} - t).$$

e^{tx} is a flat line bundle
multiplicative for the additive group law on $\mathbb{G}_a$.

$j : \mathbb{A}^1 \to \mathbb{P}^1$.

$W_t = j_* e^{tx}$ $t \neq 0 \to W_t|^*_{\{\infty\}} = 0$.

$j_! e^{tx} \xrightarrow{\simeq} W_t$ clean.

Similarly $H^\bullet_{dR}(\mathbb{A}^1, e^{tx}) = 0$. (equivariant + character nonzero $\to$ vanishing).

As t varies $\rightsquigarrow W$ in $D_{hol}(\mathbb{P}^1 \times \mathbb{G}_m)$

$\rightsquigarrow \Phi(W)$ in $D_{hol}(\mathbb{P}^1)^{\mathbb{G}_a}$.

Property i) $\Rightarrow \Phi(W)|_{\mathbb{A}^1} = \mathcal{O}_{\mathbb{A}^1}$.

$\Rightarrow \Phi(W) \simeq \mathcal{M} \oplus \delta_\infty^{\oplus r}$.

indecomposable extension of $\mathcal{O}_{\mathbb{A}^1}$.

ii) $\Rightarrow H^1(\mathbb{P}^1, \Phi(W)) = 0$.

$\Rightarrow r = 0$ and $\mathcal{M} \simeq$ tilting extension of $\mathcal{O}_{\mathbb{A}^1}$.

$\Phi(W)$

What d

characterized by $M|^!_{\{\infty\}}$ and $M|^*_{\{\infty\}}$ live in degree 0.

composition series:

$$M_1 \subset M_2 \subset M_3 = \Psi(W).$$

$$M_1 = \delta_\infty, \quad M_2/M_1 = \mathcal{O}_{\mathbb{P}^1}, \quad M_3/M_2 = \delta_\infty.$$

Verdier self dual, longest indecomposable obj of $\mathrm{Hol}(\mathbb{P}^1)^{\mathbb{G}_a}$.

injective-projective.

monodromy: $\Psi(W) \twoheadrightarrow \delta_\infty \xhookrightarrow{(M_1)} \Psi(W)$.

associated graded: $\mathfrak{sl}_2 \xrightarrow{\text{standard}} \delta_\infty$, $\xrightarrow{\text{trivial}} \mathcal{O}_{\mathbb{P}^1}$.

Main Results.

G = connected reductive group.

X = connected smooth projective curve.

Bun_G = moduli stack of G-bundles.

$B \subset G$ Borel, N nilpotent radical. $T = B/N$.

Drinfeld's compactification

$$\begin{tikzcd} \overline{\mathrm{Bun}}_B \arrow[drr, bend right, "\bar{p}"', "\text{proper}"'] & & \mathrm{Bun}_B \arrow[ll, hook'] \arrow[d, "p"] \\ & & \mathrm{Bun}_G \end{tikzcd}$$

schematic, but not proper

$G = GL_2$: $\mathrm{Bun}_B = \{ \underset{\text{line bundle}}{\mathcal{L}} \xhookrightarrow{\text{bundle embedding}} \underset{rk\,2}{\mathcal{E}} \}$.

$\overline{\mathrm{Bun}}_B = \{ \mathcal{L} \xrightarrow[\text{mono.}]{} \mathcal{E} \}$ of coh sheaves

(injective on sections)

$\Updownarrow$

(can have zero at finitely many $x \in X$).

$$\begin{tikzcd} \mathrm{Bun}_B \arrow[r, hook] \arrow[dr, "q"'] & \overline{\mathrm{Bun}_B} \arrow[d, "\bar{q}"] \\ & \mathrm{Bun}_T \end{tikzcd}$$

Fix $\omega_X^{\otimes 1/2}$

$$\rho(\omega) := 2\rho(\omega_X^{\otimes \frac{1}{2}}).$$

(T-bundle) $\quad 2\rho : \mathbb{G}_m \to T$.

$$\begin{tikzcd} \overline{\mathrm{Bun}_N^{\omega}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \overline{\mathrm{Bun}_B} \arrow[d] \\ \{\rho(\omega)\} \arrow[r] & \mathrm{Bun}_T \end{tikzcd}$$

Careful: pt/T embeds, this is not an embedding.

$\rightsquigarrow$ $\mathrm{res} : \mathrm{Bun}_N^{\omega} \to \mathbb{A}^1$.

$G = GL_2$.

$\mathrm{Bun}_{N,\omega}$ has coarse moduli space

$$H^1(X, \omega_X) \xrightarrow[tr]{} \mathbb{A}^1.$$

$j : \mathrm{Bun}_N^{\omega} \to \overline{\mathrm{Bun}_N^{\omega}}$.

$W_t := j_* \mathrm{res}^{\Delta} e^{tx}$ — coh. normalized inverse image

FGV: W_t is clean extension for $t \neq 0$.

$\rightsquigarrow W$ in $D(\overline{\mathrm{Bun}_N^{\omega}} \times \mathbb{G}_m)$

$\rightsquigarrow \Psi(W)$ on $\overline{\mathrm{Bun}_N^{\omega}}$.

We show this is tilting by describing its fibers on stratification of $\overline{\mathrm{Bun}_N^\omega}$.

$$X^{(n)} := X^n /\!\!/ \Sigma_n. \quad \underset{\underset{\sum n_\alpha \alpha.}{\parallel}}{\lambda} \in \Lambda^{pos} \rightsquigarrow X^\lambda := \prod_\alpha X^{(n\alpha)}.$$

α simple coroots

$$X^\lambda \times \mathrm{Bun}_B \xrightarrow[\text{closed}]{\text{locally}} \overline{\mathrm{Bun}_B} \quad \text{(RHS isn't smooth, LHS is).}$$

jointly surjective

$$j_\lambda : X^\lambda \times_{\mathrm{Bun}_T} \mathrm{Bun}_B \hookrightarrow \overline{\mathrm{Bun}_N^\omega}.$$

$$\downarrow r_\lambda \text{ smooth} \qquad (\mathrm{Bun}_B \to \mathrm{Bun}_T \text{ smooth}).$$

$$X^\lambda$$

$\check{\mathfrak{g}}$:= L dual Lie alg.

$\cup$ Borel

$\check{\mathfrak{n}}$

$C^\cdot(\check{\mathfrak{n}})$:= Coh. Chevalley complex. (Λ^{pos}-graded).

$\rightsquigarrow$ perverse sheaf $\Omega(\check{\mathfrak{n}})^\lambda$ on X^λ. whose !-fiber at $\sum \lambda_i x_i$ is $\bigotimes C^\cdot(\check{\mathfrak{n}})^{\lambda_i}$.

<u>Thm</u> For any $\lambda \in \Lambda^{pos}$.

$$j_\lambda^!(\Psi(\omega)) \xrightarrow{\sim} r_\lambda^\Delta \Omega(\check{\mathfrak{n}})^\lambda.$$

Why do we expect something like this? (IV).

- This is used to understand Homs in $\mathrm{Whit}^{\mathrm{ext}}$.

Fix $x \in X(k)$

$$\begin{array}{l} G(K) \hookrightarrow \mathrm{Gr}_G . \\ \cup \\ N(K) \qquad S^\lambda := N(K) \cdot t^\lambda \qquad \lambda \in \Lambda . \end{array}$$

We have $\overline{S^0} \xrightarrow{\pi} \overline{\mathrm{Bun}_N}$.

$$\tilde{\check{\mathfrak{g}}} := (\check{\mathfrak{b}} \times \check{G}) / \check{B} .$$

$$\downarrow \circlearrowleft \check{G}.$$

$$\check{\mathfrak{g}}$$

$$\tilde{\check{\mathfrak{g}}} \times_{\check{\mathfrak{g}}} \{0\} \circlearrowleft \check{G} .$$

(dg-thickening of flag variety)
(derived Springer fibers).

<u>Thm</u>. (Arkhipov-Bez, Raskin)

$$D_{\mathrm{hol}}((\mathrm{Gr}_G)^{N(K)}) \xrightarrow{\sim} \mathrm{Coh}(\tilde{\check{\mathfrak{g}}} \times_{\check{\mathfrak{g}}} \{0\})^{\check{G}} .$$

(sorta known) Conj: this upgrades to a factorizable version.

<u>Conj</u>: Under A-B equivalence,

$\pi^! \mathcal{I}(W)$ goes to the dualizing object.

$$\left(\hat{\check{\mathfrak{g}}} \times_{\check{\mathfrak{g}}} \{0\}\right)/\check{G} \longleftarrow pt/\check{B} \longrightarrow pt/\check{T}.$$

push gives Chevalley complex.

Restriction to stratum = pull-push.

Thus this agrees w/ our main theorem.

Note: should also work factorizably.