Mar 5. (Lin)

Thm 1: • Any object $\mathcal{M} \in \hat{\mathfrak{g}}_{crit}\text{-mod}^{G(\mathcal{O})}$ is supported on $\mathrm{Op}_{\check{\mathfrak{g}}}^{unr}$

• $\mathbb{V}^\lambda$ is supported on $\mathrm{Op}_{\check{\mathfrak{g}}}^{reg, \lambda}$.

Proof:

recall: $\mathrm{End}_{\hat{\mathfrak{g}}\text{-crit}}(\mathbb{M}^\lambda) = \mathfrak{Z}^{nilp, \lambda}$.

(when $\lambda + \rho \in \Lambda^+$).

First, if $\lambda \in \Lambda^+$,

$$\mathrm{Op}_{\mathfrak{g}(R)}^{nilp, \lambda} := \left\{ \nabla_0 + \sum_i t^{\langle \alpha_i, \lambda \rangle} \phi_i(t) dt f_i + \frac{q(t)dt}{t} \,\middle|\, \phi_i \in R[[t]]^\times,\ q(0) \text{ nilpotent} \right\}.$$

$$\cap$$

$\mathrm{Op}_{\mathfrak{g}}^{RS}$ $\qquad q(t) \in \mathfrak{b} \otimes R[[t]]$,

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$\mathrm{Op}_{\mathfrak{g}}^{\text{order 1 pole}}$ $\qquad$ up to gauge transformation when $\lambda+\rho$ dominant, i.e. for subset $J \subset I$,

$$j \in J \Rightarrow \langle \alpha_j, \lambda \rangle = -1.$$

otherwise $\langle \alpha_j, \lambda \rangle \geq 0$.

$J \rightsquigarrow$ parabolic. P_J. Levi m_J. $(n_J \to P_J \to m_J)$

$$Op_g^{nilp,\lambda}(R) = \{ (\text{Same formula as above}), \;\Big|\; \underbrace{\textstyle\sum \phi_j(t) f_j + q(0) \bmod n_J}_{\text{this is the residue.}} \text{ is nilpotent in } m_J \}$$

$$Op_g^{RS}(R) = \{ \nabla_0 + \textstyle\sum t^{-1} \phi_i(t) dt + q(t) dt \}.$$

$q(t)$ regular.

(First). residue map $\quad Res^{RS} : Op_g^{RS} \to g/Ad(G) = h/\!/W.$

$$\begin{array}{ccc} Op^{nilp,\lambda} & \longrightarrow & Op^{RS} \\ \downarrow & & \downarrow \\ \{ w(-\lambda-\rho) \} & \hookrightarrow & h/\!/W \end{array}$$

note: we also have <u>second</u> residue map: (takes $\mathfrak{g}(\mathcal{O})$)

i.e.

$$\begin{tikzcd} Op^{RS} \arrow[r, hook] \arrow[d] & Op^{nilp,\lambda} \arrow[d] \\ 0 \arrow[r] & n/B \end{tikzcd}$$

Now claim above amounts to

$$\mathrm{Supp}(\mathbb{M}^{\lambda}) = Op_{\check{\mathfrak{g}}}^{nilp,\lambda}$$

note if $w\cdot\lambda=\lambda'$, then $\mathrm{End}(\mathbb{M}^{\lambda}) = \mathrm{End}(\mathbb{M}^{\lambda'})$.

Since Iwahori cat are generated by those,

$\hat{\mathfrak{g}}_{crit}\text{-mod}^{I}$

get

$\sim$ supported on $\overline{\coprod_{\lambda+\rho\in\Lambda^+} Op_{\check{\mathfrak{g}}}^{nilp,\lambda}}$

$\overline{Op_{\check{\mathfrak{g}}}^{nilp,\,int}} :=$

$\exists$ Tautological bundle $\mathcal{F}_{\check{G}}$ on $Op_{\check{\mathfrak{g}}}^{nilp, int} \times \overset{\circ}{D}_x$

(actually an Local System).

which can be extended to $\check{G}$-bundle

$\mathcal{F}_{\check{G}}^{ex}$ on $Op_{\check{\mathfrak{g}}}^{nilp, int} \times D$.

s.t. the connection has a pole of order ≤ 1, nilpotent monodromy.

$\longrightarrow$ to rep language: $\mathcal{P}_{\check{G}}$: central fiber of $\mathcal{F}_{\check{G}}^{ex}$.

For $V \in Rep(\check{G})$.

- $V_{\mathcal{P}_{\check{G}}}$ is a vector bundle on $Op^{nilp, int}$.
- $N_V \in End(V_{\mathcal{P}_{\check{G}}})$ nilpotent.

s.t. they're compatible w/ $\otimes$.

lemma: • The above data can be extended canonically to $\overline{Op^{nilp, int}}$.

• Op^{unr} is the locus where $\overline{N_V}$ vanishes.

For part 1 of thm 1, it remains to prove that for $V \in Rep(\check{G})$, and $M \in \hat{\mathfrak{g}}\text{-mod}^{G(O)}$

extend vector bundle $\overline{V_{\mathfrak{z}_{\check{G}}}} \underset{\mathfrak{z}^{nilp,int}}{\otimes} M$ has trivial $\overline{N_V}$ action

in fact, it's $Sat_V \star M$.

Comes from **Lemma**: $M \in \hat{\mathfrak{g}}_{crit}\text{-mod}^{I}$.

$$\overline{V} \underset{\mathfrak{z}^{nilp,int}}{\otimes} M = Z \underset{I}{\star} M.$$

construction of some nearby cycle

$Z_I \in D(Fl_G)_{crit}\text{-mod}^{I}$.

$P_!(Z_V) = Sat_V$.

$P: Fl_G \to Gr_G$.

$N_{Z_V} \in End(Z_V)$.

and $P_!(N_{Z_V}) = 0$.

This concludes part 1.

For affine Weyl, use

$$Op^{unr} \cap Op^{nilp,\lambda} = Op^{reg,\lambda}.$$

Recall Drinfeld-Sokolov reduction: $\mathfrak{Z} \triangleq \mathrm{Center}(\hat{\mathfrak{g}}_{crit}^{-ur})$

$$D^{+}(\hat{\mathfrak{g}}_{crit}\text{-mod}) \longrightarrow D(Vect) \text{ in fact } D(\mathfrak{Z}\text{-mod})$$

$$M \longmapsto C^{\frac{\infty}{2}}(n((t)), n[[t]], M \otimes \chi).$$

in particular,

$$\Phi : D^{+}(\hat{\mathfrak{g}}_{crit}\text{-mod})^{L^+G} \longrightarrow D(\mathfrak{Z}^{unr}\text{-mod})$$

Thm 2: Φ is exact.

(Proof omitted).

Thm 3:

1) $\mathfrak{Z}_{\mathfrak{g}}^{\lambda, reg} \longrightarrow \mathrm{End}(\mathbb{V}^{\lambda})$ is an iso.

2) $\mathrm{End}_{\hat{\mathfrak{g}}\text{-crit}}(\mathbb{V}^{\lambda}) \longrightarrow (\mathbb{V}^{\lambda})^{\mathrm{Lie}(I^{u})}_{\lambda} = (\mathbb{V}^{\lambda})^{G(0)}_{\lambda}$ (well defined by thm 1)

$f \longmapsto f(\mathbb{1})$

$\hookrightarrow (\mathbb{V}^{\lambda})^{\mathrm{Lie}(I_u)} \xrightarrow{\quad} \Phi(\mathbb{V}^{\lambda})$ (by def of Φ)

is an isomorphism.

3) $\mathbb{V}^{\lambda}$ is a proj $\mathfrak{Z}^{reg,\lambda}$-mod.

Cor: image of Weyl under Φ is $\mathfrak{Z}_{\mathfrak{g}}^{\lambda, reg}$

since Weyl generate whole cat,

it remains to check Hom-compatibility.

Pf:

Step 1: $\mathfrak{Z}_{\mathfrak{g}}^{reg,\lambda} \xrightarrow{(1)+(2)} \Phi(\mathbb{V}^{\lambda})$ is injective.

Recall $Op_{\check{G}}(D^{\times})$ has a Poisson structure.

$\Rightarrow \Omega^1(Op_{\check{G}}(D^{\times}))$ is a Lie algebroid.

on $Op_{\check{G}}(D^{\times})$.

Explicit description of corresponding groupoid:

Isomonodromy opers: $\mathrm{Isom}_{Op_{\mathfrak{g}}}(R) =$

$\{ (\chi, \chi', \rho) \mid \chi, \chi' \in Op(D^\times), \rho :$ iso between them as local systems$\}$

$$\begin{tikzcd} Isom \arrow[r] \arrow[dr] & Op(D^\times) \\ & Op(D^\times). \end{tikzcd}$$

Lemma:

$Op_{\check{g}}^{\times, reg} \subset Op(D^\times)$ is co-isotropic.

i.e. $\{I, I\} \subset I$ for I the ideal.

$N^*_{Op_g^{reg, \lambda}}$ is an Lie algebroid on $Op^{reg, \lambda}$.

General Poisson theory

$\rightsquigarrow \mathfrak{Z}_G^{reg, \lambda}$ is an irreducible.

$N^*_{Op_{\check{G}}^{reg, \lambda}}$ - module.

Recall $\mathbb{V}_{crit}^{\lambda} \in QCoh(Op^{reg, \lambda})$

and is also N^* - mod by def.

Since Φ is functorial, $\mathcal{N}^*$ also acts on $\Phi(\mathbb{V}_{\widehat{\mathrm{crit}}})$.

Claim: $(1) \to (2)$ is an $\mathcal{N}^*$-mod-morphism.

suffices to prove it's nonzero since LHS is irred.

But $1 \mapsto 1$, so nonzero.

Step 2: (Dimension Comparing).

Recall $\mathrm{Aut}(D) \circlearrowright$ everything.

$L_0 = -t\partial_t \in \mathrm{Lie}(\mathrm{Aut}(D)) \circlearrowright$ everything.

it gives a grading on everything.

and check that each graded piece has the same dimension.

Thus $(1) \to (2)$ is iso.

Step 3:

recall
$$\begin{tikzcd} \mathfrak{z}^{\lambda\text{-reg}} \arrow[r] & \mathrm{End}(\mathbb{V}^{\lambda}) \arrow[r, leftrightarrow] \arrow[rr, bend left, "\cong \varphi"] & (\mathbb{V}^{\lambda})_{\lambda}^{\mathrm{Lie}(I_u)} \arrow[r, "h"] & \Phi(V^{\lambda}) \end{tikzcd}$$

(1)+(2): it remains to check h or φ injective.

Suppose $E \in \mathrm{End}(\mathbb{V}^{\lambda})$ is s.t.

$$\varphi(E) = 0.$$

$$\varphi(E) = h(E(\mathbb{1})) = \Phi(E)(h(\mathbb{1})).$$

$h(\mathbb{1}) \neq 0$ since whole map is iso

$$\Rightarrow \Phi(E) = 0.$$

since Φ is exact.

$$0 \to K \to \mathbb{V}^{\lambda} \xrightarrow{E} \mathbb{V}^{\lambda}$$

$$\Rightarrow \Phi(K) = \Phi(\mathbb{V}^{\lambda}).$$

note $E \neq 0$, K does not contain $\mathbb{C}\mathbb{1}$.

then under the grading given by L_0, K belongs to to grading part, so $\Phi(K)$ is as well.

$\Rightarrow \Leftarrow$

Part (3) uses Wakimoto, skipped for now.

Now fully faithfulness:

Thm 4: $R\mathrm{Hom}_{\hat{\mathfrak{g}}_{crit}^{L+G}}(\mathbb{V}^{\lambda}, \mathbb{V}^{\lambda})$

$$= R\mathrm{Hom}_{\mathfrak{Z}_{\mathfrak{g}}^{unr}}(\mathfrak{Z}_{\mathfrak{g}}^{reg,\lambda}, \mathfrak{Z}_{\mathfrak{g}}^{reg,\lambda}).$$

(I'll try to cover this next week).

($\mathbb{V}^{0}$ given in Frenkel - Teleman).

(essential surjectivity is rather easy after this).