$\mathrm{LocSys}(D^\times)$

"moduli of loc sys on the punctured disc"

$d t + \mathfrak{g}((t)) / G((t))$

$g \curvearrowright x \longrightarrow g x g^{-1} - g^{-1} dg$.

Note: we do not sheafify.

e.g. $G = \mathbb{G}_a$, get $\mathfrak{g}/G$. (True for G unipotent).

$dt + \mathfrak{g}_a((t)) / \mathbb{C}((t))$.

$g \curvearrowright x \Rightarrow x - dg$.

2) $G = \mathbb{G}_m$

get $\mathbb{A}^{\infty}_{dR} \times B\mathbb{G}_m \times \mathbb{C}/\mathbb{Z}$.

$\mathrm{Hom}(S, \mathbb{G}_m((t))) = S((t))^\times$

$= a t^i + b(t) t^{i+1} + C(t)$

a invertible

$b \in S[[t]]$. $C \in S^{nil}((t))$ nilpotent.

$$G_m((t)) = \mathbb{Z} \times G_m \times \prod_{\infty} A^1.$$

$$G_m[[t]] = G_m \times \prod_{\infty} A^1.$$

$g \curvearrowright X$: G_m does nothing

$\mathbb{Z}$: shifts deg

$\prod A^1$: changes deg >0. component freely w/ no stab.

A_0^∞ : kills the formal neighborhood of neg part.

$$A^\infty \times \prod_{\infty} A^1 \times \mathbb{C} / \mathbb{Z} \times G_m \times \prod_{\infty} A^1 \times \hat{A}_0^\infty.$$

$$= A_{dR}^\infty \times \mathbb{C}/\mathbb{Z} \times BG_m.$$

3) $G = GL_2$. (or $G_m \ltimes G_a$).

$$d t + \begin{bmatrix} -\frac{n}{t} & t^{n-1} \\ 0 & 0 \end{bmatrix}.$$

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \rightsquigarrow \begin{pmatrix} a'(t) - \frac{n a(t)}{t} + t^{n-1} b(t) \\ b'(t) \end{pmatrix}$$

(finitely many)

Hope: if I know the first terms of my connection, I know my connection.

e.g. $dt + a(t)$

$a(t) \in \mathbb{C}[[t]]$

is always trivial.

(then, $t^{-r}\mathfrak{g}[[t]] \longrightarrow t^{-r}\mathfrak{g}[[t]]/G[[t]]$

factors through some $t^{-r}\mathfrak{g}[[t]]/t^{r+s}$

then things are locally finite type).

This does not hold.

consider the example above.

i.e. $\mathrm{LocSys}(\mathring{D})$ not (locally) finite type.

$$t^{-r}\mathfrak{g}[[t]] / K_{r+s}\,(t^{r+s}\mathfrak{g}[[t]]).$$

congruence subgroup.

$$\downarrow$$

fibers are Artin stacks for large S. (Thm, Sam).

$$t^{s}\mathfrak{g}[[t]] / t^{-r}\mathfrak{g}[[t]].$$

Slopes.

Again, goal: classify connections on $D^{\times}$. (GL_n)

Fix connection.

$$\lim_{n\to\infty} \frac{\dim\left(\left((t\nabla_t)^n \mathbb{C}[[t]]^{\oplus r}\right) / \mathbb{C}[[t]]^{\oplus r}\right)}{n^r}$$

Lemma. This is gauge invariant.

$\Leftarrow$ Can take any lattice W in the stalk of instead of $\mathbb{C}[[t]]^{\oplus r}$.

Lemma RS $\Leftrightarrow$ slope 0.

$d t + t^{-n}$ has slope $n-1$.

<u>Thm</u>.

slope exists and is a rational # w/ denom. $\leq r$.

Pf: cyclic vector thm (Deligne).

$\exists$ a vector v s.t.

$v, \nabla v, \ldots \nabla^{r-1} v$.

are $\mathbb{C}((t))$ linearly independent.

assume for any v, those are linearly dependent.

chose f not in span.

Consider $v' = v + t^k f$ instead.

<u>Corollary</u> every connection is equivalent to the following form

$$\partial_t + \begin{bmatrix} 1 & & 0 & a_1 \\ & 1 & & a_2 \\ 0 & & 1 \ddots & \vdots \\ & & & a_5 \end{bmatrix}.$$

$$\text{slope} = \min\left(0, -\frac{v(a_i)}{r_i}\right).$$
(?)

$$= \cdots (??)$$

(something happened).

can bring into some form.
which induces Sam's theorem.

1. Artinness $\leftarrow\!\sim$ Fredholmness of exth.

2. Proof that
$$t^{-r}\mathfrak{g}[[t]]/G(0) \text{ is } \text{ind-affine}.$$

3. Consequences for LocSys.

Recall $\mathbb{A}^1, \mathbb{A}^2 \cdots$ all 1-affine

but $\mathbb{A}^\infty$ isn't

(cotangent complex is infinite)

but does give Loc fully faithful.

(i.e. $\Gamma \circ \mathrm{Loc}_{\mathbb{A}^\infty} = \mathrm{id}$).

$\Gamma = \lim \Gamma_{A_i}$.

$\Gamma \circ \mathrm{Loc}(C) = \lim C \otimes_{QCoh(\mathbb{A}^\infty)} QCoh(\mathbb{A}^i)$.

lim across pullback + embedding $\rightsquigarrow$ colimit

$\rightsquigarrow$ colimit, good.

Do same for $\mathfrak{g}((t))/G(O)$

upgrade to $G(K)$

using $G(K)/G(O)$ ind-proper.

so QCoh rigid

Sam: use $\underbrace{t^{-r}\mathfrak{g}[[t]]}$ is $G(O)$-tame.

i.e. $\mathcal{C} = \mathrm{QCoh}(\uparrow)$.

$$\mathcal{C}_{G(O),w} \longrightarrow \mathcal{C}^{G(O),w}$$

is equivalence.

$\Downarrow$

implies $t^{-r}\mathfrak{g}[[t]]/G(O)$ is 1-affine.

Remark: $QCoh(X) \rightleftarrows QCoh(Y)$ monadic

$\Updownarrow$

$X \longrightarrow Y$ descent.

$\Updownarrow$

$QCoh(X)$ has a geometric realization in terms of $QCoh(Y)$.

in this case,

$$\begin{tikzcd} t^{-r}\mathfrak{g}[[t]] \arrow[d, "\pi^*"] \\ t^{-r}\mathfrak{g}[[t]]/G(0). \arrow[u, bend left, "\pi_?"] \end{tikzcd}$$