1. Affineness (Yuval)

Let $\mathcal{Y} \in \text{PreStk}$.

Define cat $\text{ShvCat}(\mathcal{Y})$ as:

obj: $\mathcal{C} \in \text{ShvCat}(\mathcal{Y})$ is the following data:

$\forall S \in \text{DGSch}^{\text{aff}}_{/\mathcal{Y}}$, $\Gamma(S, \mathcal{C})$, which is a module cat over $\text{QCoh}(S)$.

$\forall S_2 \to S_1$.

$$\Gamma(S_1, \mathcal{C}) \otimes_{\text{QCoh}(S_1)} \text{QCoh}(S_2) \simeq \Gamma(S_2, \mathcal{C}).$$

then for $\mathcal{Y}_1 \xrightarrow{\pi} \mathcal{Y}_2$, ~~nice enough~~ get adjunction

$$\text{ShvCat}(\mathcal{Y}_1) \underset{\pi^*}{\overset{\pi_*}{\rightleftarrows}} \text{ShvCat}(\mathcal{Y}_2).$$

Test. What is $\text{ShvCat}(pt)$?

Answer: DGCat.

Example: $\mathcal{Y} \in \text{PreStk}$. $\mathcal{C} \in \text{ShvCat}(\mathcal{Y})$.

$\text{QCoh}_{/\mathcal{Y}} \in \text{ShvCat}(\mathcal{Y})$.

Global Sections.

$\mathcal{Y} \in \text{PreStk}$. $\mathcal{C} \in \text{ShvCat}(\mathcal{Y})$.

$$\Gamma(-, \mathcal{C}) : (\text{DGSch}^{\text{aff}}_{/\mathcal{Y}})^{op} \to \text{~~DGCat~~ } \text{QCoh}(\mathcal{Y})\text{-mod}.$$

RKE $\downarrow$ PreStk $\nearrow$

So we have two functors:

$$\Gamma_{\mathcal{Y}} : \text{ShvCat}(\mathcal{Y}) \underset{\longrightarrow}{\longleftarrow} \text{QCoh}(\mathcal{Y})\text{-mod} : \text{Loc}_{\mathcal{Y}} .$$

(note that this graph is mirrored).

$\text{Loc}_{\mathcal{Y}}$ is the following:

$$\begin{array}{c} \mathcal{C} \\ \cap \\ \text{QCoh}(\mathcal{Y})\text{-mod} \end{array} \rightsquigarrow \left(\begin{array}{c} S \to \Gamma(S, \text{Loc}_{\mathcal{Y}}(\mathcal{C})) = \text{QCoh}(S) \otimes_{\text{QCoh}(\mathcal{Y})} \mathcal{C} \\ \text{- - -} \end{array} \right).$$

When is this an equivalence?

<u>Def</u>: when $\mathcal{Y}$ is 1-affine.

Note that affine DG schemes are tautologically 1-affine.

<u>Grand Thm:</u>

$\mathcal{Y} \in$ ~~[illegible]~~ AlgSpace.

1) finite type

2) $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$.
qc, schematic, quasi-affine.

3) $\text{QCoh}(\mathcal{Y})$ dualizable.

(conditions 1)–3): $\triangleq$ $\mathcal{Y}$ passable.)

then $\mathcal{Y}$ is 1-affine.

Note: any quasi-compact quasi-separated algebraic space is passable.

Proof of GT:

Claim: under such conditions, $\exists$ diagram:

$$\begin{array}{ccc} U' & \xrightarrow{j'} & U \\ {\scriptstyle f'}\downarrow & \ulcorner & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

s.t. j, j' open, U, Y', U' 1-affine.

f, f' étale, f is 1-1 over $Y' - Y$.

Proof of claim: By induction: $\emptyset = Y_0 \subset \cdots Y_k \subset = Y$ open stratification

$$\begin{array}{c} Y_k \\ \uparrow {\scriptstyle f_k} \\ U_k. \end{array}$$

U_k quasi-affine scheme.

quasi compact. f_u étale. $Y' = Y_{k-1}$, $U' = U_{k-1}$, $U = U_k$.

By induction NTS: can take U, U' affine. (Skipped).

Proceed to main thm:

Claim: For $C \in QCoh(Y)$-mod. (Nisnevich Descent)

$$\begin{array}{ccc} QCoh(U') \otimes_{QCoh(Y)} C \ \ulcorner & \longleftarrow & QCoh(U) \otimes_{QCoh(Y)} C \\ \uparrow & & \uparrow \\ C \otimes_{QCoh(Y)} QCoh(Y') & \longleftarrow & C \end{array}$$