Hyperspeed Overview of [B].

We ~~continue~~ inherit notations from KC's talk on [AB]. Today our goal is to construct monoidal equivalences

> Notation: $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} =: St$, $\tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathcal{N}} =: St^d$
> $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathcal{N}} =: St'$

$$D(\check{I}\backslash \check{G}(k)/\check{I}) \simeq \mathrm{IndCoh}((\tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathcal{N}})/G)$$

i.e. generated by pullback from $D_{\check{I}^\circ \check{I}}$. $\to$ $D_{mon}(\check{I}^\circ \backslash \check{G}(k)/\check{I}^\circ) \simeq \mathrm{IndCoh}_{\tilde{\mathcal{N}}}((\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}})/G)$

set-theoretic support

as well as their module category

$$D(\check{I}^\circ\backslash \check{G}(k)/\check{I}) \simeq \mathrm{IndCoh}((\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathcal{N}})/G)$$

where we remind that $\tilde{\mathcal{N}}$ is the Springer resolution and $\tilde{\mathfrak{g}}$ is the Grothendieck-Springer resolution.

In his senior thesis, James Tao demonstrated how to streamline a lot of [B] using higher category theory. Much of this talk will be copy-paste from his thesis.

<u>Recalling AB</u>

Recall that we got a functor $\Phi_{diag}: \mathrm{Coh}(G\backslash \tilde{\mathcal{N}}) = \mathrm{Coh}(B\backslash \mathfrak{n}) \longrightarrow D_{mod}(I\backslash G(k)/I)$ (loc. comp.) via the following procedure:

- get $\mathrm{Coh}(pt/T)$ from Wakimoto.
- get $\mathrm{Coh}(pt/G)$ via central sheaves.

} Drinfeld-Plucker to get $\mathrm{Coh}(pt/B)$ (using Wakimoto filtration of central sheaves)

$\downarrow$ upgrade to $\mathrm{Coh}(\mathfrak{g}/G)$ via log of monodromy

then combine.

(We also get $\Phi_{IW}: \mathrm{Coh}(G\backslash \tilde{\mathcal{N}}) \xrightarrow{\sim} D_{mod}(\check{Fl})^{(\check{I}_a, \chi), lc}$ which intertwines the action on both sides.)

I'll be forgetting these very often...

The plan of attack is as follows:

1) Results in derived geometry yields that

$$\mathrm{Perf}(St^d/G) \simeq \mathrm{Perf}(\tilde{\mathcal{N}}/G) \otimes_{\mathrm{Perf}(\tilde{\mathfrak{g}}/G)} \mathrm{Perf}(\tilde{\mathcal{N}}/G).$$

So a map out of it can be obtained via a $\tilde{\mathfrak{g}}/G$-compatible map out of $\mathrm{Perf}(\tilde{\mathcal{N}}/G)$.

In particular We obtain the by letting $\tilde{\mathcal{N}}/G$ acting on both sides via Φ_{diag} on a tilting object Ξ.

a map to $D(\check{Fl})^{\check{I}}$

2) Given any functor $\Phi: C \to D$ of DGCat, we can always form the ind-adjoint.

$$D \xrightarrow{\text{Yoneda}} \mathrm{Funct}(D^{op}, \mathrm{Vect}) \xrightarrow{\Phi^{op}} \mathrm{Funct}(C^{op}, \mathrm{Vect}).$$

which gives a map $\Phi: D(\check{Fl})^{\check{I}} \longrightarrow \mathrm{Ind}(\mathrm{Perf}(St^{d}/G)) \simeq \mathrm{QCoh}(St^{d}/G).$

Claim: explicit cohomological vanishing can guarantee that it's an iso onto Coh.

2.5) One shows that conditions above can be rephrased purely in terms of Φ.

3) To check the monoidal/module category claims we will need to invoke some compatibilities with Φ_{Iw}.
to construct another functor Ψ': (automorphic) $\to$ (spectral) and show it coincides with Φ above.

Remark on Free-monodromic Sheaves

Recent development in higher algebra now allows us to largely bypass the use of free-monodromic objects in this story. For completeness, however, we include a brief explanation of what's going on.

The difficulty lies in accessing $\mathrm{Coh}(\tilde{N} \times_{g} \tilde{N})^{G}$, which involves a derived tensor product. To get around this issue, one can try to consider $\underset{St \times_{t} \{0\} =}{\tilde{N} \times_{g} \tilde{N}} \xrightarrow{\iota} \hat{St} :=$ formal completion of St over $N \subseteq g$.

t^2

Claim: $\iota^{*}_{\mathrm{IndCoh}}$ exists. (Standard Koszul duality: QCoh pullback is equivalent to $- \otimes_{U(t^2)} k$; now k has a finite resolution).

Corollary: $(\iota^{*}_{\mathrm{IndCoh}}, \iota_{*\,\mathrm{IndCoh}})$ restricts to an adjoint pair (ι^{*}, ι_{*}) between Coh.

Since ι_{*} is conservative, BBL says we can express $\mathrm{Coh}(\tilde{N} \times_{g} \tilde{N})^{G}$ as $(\iota_{*} \circ \iota^{*})$-mod in $\mathrm{Coh}(\hat{St})^{G}$; the latter does not need derived geometry to express.

By same reason, ι^{*} of $\mathrm{Coh}(\hat{St})^{G}$ generates the category, so if we can express $\mathrm{Coh}(\hat{St})^{G}$ on the dual side we'd be able to obtain the desired iso by matching monads.

Now since $\hat{St}$ is an indscheme, its Coh contains some "pro-objects" (e.g. structure sheaf) and these turn out to be the pro-monodromic objects on the dual side.

More precisely, let $\hat{D}$ be a certain pro-completion of $D(\check{I}_0 \backslash \check{G}(K) / \check{I}_0)$ at the origin. Cf [B-Y A-3-1].

Then we have $\hat{D} \simeq \mathrm{IndCoh}(\hat{St}/G)$, which is an upgrade of the statement regarding D_{mon}. ~~AB~~ This can be established by upgrading entire [AB] to this setting.

*: let $X \to Y$ be a T-torsor, then we get the Atiyah bundle

$$0 \to \mathrm{Lie}(T) \otimes \mathcal{O}_Y \to At \to \Theta_Y \to 0.$$

Let D be the $\mathcal{O}_Y$-algebra generated by At. then $D_{mon}(X) = \{D\text{-modules on which } \mathrm{Lie}(T) \text{ acts nilpotently}\}$.

Since $\mathrm{Lie}(T)$ is central, we can consider

$$\hat{D} := \widehat{\mathrm{Sym}(\mathrm{Lie}\,T)} \underset{\mathrm{Sym}(\mathrm{Lie}\,T)}{\otimes} D, \text{ where } \wedge \text{ is completion wrt ideal } (\mathrm{Lie}\,T).$$

and $\hat{D}$ above is roughly (how?) the category of $\hat{D}$-modules.

The monoid on $\hat{D}$ is then the action by log of monodromy, and one proceeds to match them.

We shall <u>not</u> go down this path in this talk. of $\hat{U}(t^2)$

Tilting Objects

In any highest weight category C, an object T is <u>tilting</u> if it admits both a standard flag and a costandard flag.

<u>Prop.</u> (Tilting Exercises)

- ~~Tilting~~ Indecomposable tilting objects are in bijection with the standards.
- They generate C.
- For any two tilting objects T_1, T_2, $\mathrm{Ext}^n(T_1, T_2) = 0 \quad \forall n > 0$.

$\Rightarrow \underset{\text{bounded homotopy}}{K^b(T)} = D^b(C)$. ← This allows [B] to bypass difficulties with derived categories. We shall not worry about this aspect.

By this proposition, for any $w \in W_{aff}$ we have a corresponding tilting object $T_w \in D(\tilde{Fl})^{I}$.

Our main player is $\Xi := T_{w_0}$. It is also the projective cover of Δ_0, ~~[illegible]~~ known as the "big projective" in category $\mathcal{O}$.

Let $Av_{I\to Iw}: D(\check{Fl})^{\check{I}} \to D(\check{Fl})^{(\check{I}_0^-, \chi)}$ and $Av_{Iw\to I_\phi}: D(\check{Fl})^{(\check{I}_0^-, \chi)} \to D(\check{Fl})^{\check{I}}$

be the !-averaging functors.

<u>Prop</u>. (Right convolution with Ξ) $= (Av_{Iw\to I_\phi} \circ Av_{I_\phi \to Iw})$.

(Compare in category $\mathcal{O}$: $(- * \Xi) = (T_{\rho\to\lambda} \circ T_{\lambda \to -\rho})$.) (left conv.) (see back side)

(We might prove this if we have time.)

<u>Construction of Φ_{perf}: $Perf(St^d/G) \to D(\check{Fl})^{\check{I}}$</u>

A derived stack X with affine diagonal is called <u>perfect</u> if $QCoh(X) = Ind\,Perf(X)$.

Equivalently, this means $\mathcal{O}$ is compact and $QCoh$ is compactly generated.

e.g. - any derived qc (quasi-compact) scheme.

- (quasi-projective scheme)/G, G affine

$\Rightarrow$ $\tilde{N}/G$, g/G are perfect.

<u>[BZFN]</u> For $X \to S$, $Y \to S$ maps of perfect stacks, $Perf(X \times_S Y) \simeq Perf(X) \otimes_{Perf(S)} Perf(Y)$.

Thus $Perf(St^d/G) \simeq Perf(\tilde{N}/G) \otimes_{Perf(g/G)} Perf(\tilde{N}/G)$.

So to get Φ_{perf}, suffices to map from $Perf(\tilde{N}/G)^2$ in a way compatible with $Perf(g/G)$-action. Namely, we let both sides map to $D(\check{Fl})^{\check{I}}$ then act on Ξ via convolution. Compatibility was explained in KC's talk.

<u>Prop</u>. Φ_{perf} is fully faithful. ([B, 6.2])

<u>Pf sketch</u>. Suffices to check on generators. Conv by Wakimoto/central sheaves have obvious inverses, so we reduce to check

$$Hom(\mathcal{O}_{St^d}(0,\mu), V^\bullet \otimes \mathcal{O}_{St^d}(\lambda^\bullet, 0)) \simeq Hom(\Xi * J_\mu, J_{\lambda^\bullet} * Z_{V^\bullet} * \Xi).$$

Left Conv: Under the equivalence $\Phi_{Iw} : Coh(\tilde{N}/G) \xrightarrow{\sim} Dmod(\check{Fl})^{(\check{I}_0, \chi), lc}$

convolution from left by Ξ corresponds to $p^* p_*$, where $p: \tilde{N}/G \to \mathfrak{g}/G$ is the projection.

Now we use the conv. description above:

$RHS = Hom(Av_{I_0 \to Iw}(\Xi * J_\mu), Av_{I_0 \to Iw}(J_\lambda * Z_V))$ (we use the fact that $Av_{Iw \to I_0} \circ Av_{I_0 \to Iw} = Av^*_{Iw \to I_0} \circ Av_{I_0 \to Iw}$)

$= Hom_{Coh(\tilde{N}/G)}(p^* p_* \mathcal{O}(\mu), \mathcal{O}(\lambda) \otimes V)$ (since $\Phi_{Iw} = Av_{I_0 \to Iw} \circ \Phi_{diag}$).

$= Hom_{Coh(\tilde{N}/G)}(p_{s,2 *} p_{s,1}^* \mathcal{O}(\mu), \mathcal{O}(\lambda) \otimes V))$

$p_{s_{1,2}}: St^d/G \to \tilde{N}/G$.

$= Hom_{Coh}(p_{s,1}^* \mathcal{O}(\mu), p_2^! (\mathcal{O}(\lambda) \otimes V))$ #.

($p_{s_i}^* = p_{s_i}^!$)
b/c $\omega = \mathcal{O}[d]$ in both source & target.

Construction and iso of Φ

As explained in introduction, we always have the ind-right adjoint

$$\Psi : Dmod(\check{Fl})^{\check{I}, lc} \to Ind\,Perf \simeq QCoh(St^d/G).$$

The following allows us to say when it ~~[illegible]~~ is an iso to Coh.

Prop. Let $\Phi : Perf(Z/H) \to C$ ← finite type and $\Psi: C \to Ind(Perf(Z/H)) \simeq QCoh(Z/H)$ be as above. Assume Φ fully faithful and C has bounded t-structure.

Further assume:

1) Ψ lands in Coh.

2) Φ has finite cohomological amplitude.

3). $\exists\, d > 0$ s.t. $\forall M \in C$, $H^{d'}(\Psi(M)) = 0 \ \forall |d'| \leq d \Rightarrow H^0(M) = 0$.

Then Ψ factors through $C \simeq Coh(Z/H)$.

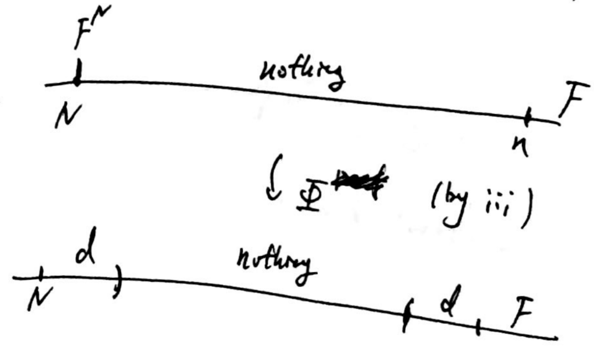
Let's explain why this can be true. [redacted], for any $F \in \mathrm{Coh}^{\geq N,\, N \ll 0}_{(H)}$ we can find

$$(F^{\leq N} \to F^{\mathrm{perf}} \to F \to \cdot)$$ (take $\mathcal{E}$ locally free $\twoheadrightarrow F$ aka Hilbert syzygy, then stupidly truncate). (*) $\to$

where F^{perf} is in Perf and $F^{\leq N}$ is ~~sufficiently negative degrees~~ concentrated in deg N.

Since Ψ lands in Coh, $\mathrm{Hom}(\Psi(A), \Psi(B)) \simeq \mathrm{Hom}(\mathcal{E}, \Psi(B))$ for some good perf-approximation $\mathcal{E}$ of $\Psi(A)$. Φ being fully faithful implies this $\simeq \mathrm{Hom}(\Phi(\mathcal{E}), B)$. At the same time, $\mathcal{E} \to \Psi(A) \leftrightsquigarrow \Phi(\mathcal{E}) \to A$, which is a good approximation of A itself by iii), so actually $\mathrm{Hom}(A, B) = \mathrm{Hom}(\Phi(\mathcal{E}), B)$. So fully faithful.

To get ess. surj., choose $N \ll 0$, then by coh. assumption, F^{perf} looks like this and $F^{N} \to F^{\mathrm{perf}} \to F$ $(F \in \mathrm{Coh}^{\geq n})$



Then $\Phi(F^{\mathrm{perf}})$ splits into τ_{left} and τ_{right}, whence $\Psi(\tau_{\mathrm{right}}\Phi(F^{\mathrm{perf}})) \simeq F$ b/c Ψ has bounded cohomological amplitude.

But how do we use this?

The criterion above concerns cohomological vanishing in $\mathrm{Coh}(Z/H)$, but how do we detect those?

This turns out to be fairly explicit if Z is projective over an affine base, because we can reduce to affine, for which the criterion is explicit: it amounts to computation of $\mathrm{Hom}(\mathcal{O}, F[i])$.

Since $\mathrm{oblv}: \mathrm{Coh}(Z/H) \to \mathrm{Coh}(Z)$ is t-exact, it suffices to compute

$$\mathrm{Hom}_Z(\mathcal{O}_Z, \mathrm{oblv}\, F) \overset{\text{(standard pull-push)}}{=} \mathrm{Hom}_{Z/H}(\mathcal{O}_{Z/H}, F \otimes_k \mathcal{O}_H) = \bigoplus_{\lambda \in \Lambda^+} \mathrm{Hom}_{Z/H}(\mathcal{O}_{Z/H}, F \otimes_k V_\lambda) \otimes_k V_\lambda^*.$$

This reduces us to the quasi-affine case. Writing $Z \simeq \mathrm{Proj}\, A$ and let $\bar{Z} = \mathrm{Spec}\, A/\mathbb{G}_m$. We have an open embedding $j: Z \hookrightarrow \bar{Z}$. The following is a derived version of Serre vanishing thm:

Prop 1) For $F \in \mathrm{Coh}(Z)$, for $k \gg 0$, $\forall d$, $H^d\, \mathrm{gr}_{\geq k} j_* F = 0 \iff H^d F = 0$.

2) for any k, $\forall F \in \mathrm{Coh}(Z)$,

$$F \in \mathrm{Coh}(Z) \iff \mathrm{gr}_{\geq k} j_* F \in \mathrm{Coh}(\bar{Z}).$$

Then relevant cohomology vanishing is given by considering

$$\Gamma(\operatorname{Spec} A, \operatorname{gr}_{\geq k} j_* \operatorname{oblv} \Xi(M))$$

$$= \bigoplus_{n \geq k} \Gamma(Z, (\operatorname{oblv} \Xi(M))(n))$$

$$= \bigoplus_{n \geq k} \operatorname{Hom}(\mathcal{O}_{Z/H}(-n), \Xi(M) \otimes \mathcal{O}_H)$$

$$= \bigoplus_{\substack{n \geq k \\ \lambda \in \Lambda^+}} \operatorname{Hom}(\mathcal{O}_{Z/H}(-n) \otimes V_\lambda, \Xi(M)) \otimes V_\lambda = \bigoplus_{\substack{n \geq k \\ \lambda \in \Lambda^+}} \operatorname{Hom}_{D(\check{Fl})^{\check{I}}}(\Xi_{perf}(\mathcal{O}_{Z/H}(-n) \otimes V_\lambda), M) \otimes V_\lambda.$$

~~from~~ which ~~the~~ now follows from our knowledge about Ξ_{diag}.

Putting everything above together, we ~~can~~ have obtained an equivalence $D(\check{Fl})^{\check{I}} \simeq \operatorname{IndCoh}(St^d/G)$ as plain DG categories. Let $\Xi: \operatorname{Coh}(St^d/G) \xrightarrow{\sim} D(\check{Fl})^{\check{I}, lc}$ be the opposite direction.

<u>Recovering the monoidal structure</u>

It turns out there's a much faster way of obtaining Ξ. Namely, one can prove w/o much difficulty that $\operatorname{Perf}(\tilde{N}/G)$ is ~~dua~~ self-dual as $\operatorname{Perf}(\mathfrak{g}/G)$-module, and we get (using $\tilde{N} \to \mathfrak{g}$ proper sheaves).

$$\operatorname{Coh}(St^d/G) \simeq \operatorname{Perf}(\tilde{N}/G) \otimes_{\operatorname{Perf}(\mathfrak{g}/G)} \operatorname{Perf}(\tilde{N}/G) \simeq \operatorname{Hom}_{\operatorname{Perf}(\mathfrak{g}/G)}(\operatorname{Perf}(\tilde{N}/G), \operatorname{Perf}(\tilde{N}/G))$$

which is moreover a <u>monoidal</u> equivalence.

Now observe that $\Xi_{Iw}: \operatorname{Perf}(\tilde{N}/G) \simeq \operatorname{Coh}(\tilde{N}/G) \xrightarrow{\Xi_{diag}} D(\check{Fl})^{\check{I}} \xrightarrow{\operatorname{Av}_{I_0 \to Iw}} D(\check{Fl})^{(\check{I}_0, \chi)}$ can be upgraded to a map of $\operatorname{perf}(\mathfrak{g}/G)$-module categories. At the same time, action of $D(\check{Fl})^{\check{I}}$ on $D(\check{Fl})^{(\check{I}_0, \chi)}$ ~~to~~ commutes with $\operatorname{Perf}(\mathfrak{g}/G)$ action via central sheaves. Thus the action map provides another functor

$$\Xi': D(\check{Fl})^{\check{I}, lc} \longrightarrow \operatorname{Coh}(St^d/G)$$

(It's not clear how to show directly this is an equivalence.)

Our job now is to show that $\Xi' \simeq \Xi$ as plain functors.

Recall the inclusion of a full subcategory $C_0 \overset{i}{\hookrightarrow} C$ is called <u>dense</u> if $LKE_i\, i = Id_C$.
Typical example of this is the Yoneda embedding.

<u>Prop</u>. For X as discussed above (having the approximation property), $Perf \overset{i}{\hookrightarrow} Coh$ is dense.
(This is a repackaging of the approximation property).

Then consider

$$\begin{tikzcd} Perf(St^d/G) \arrow[r, hook, "i"] \arrow[d, hook, "i"] & Coh(St^d/G) \arrow[r, "act", "\sim"'] & End_{Perf(\mathfrak{g}/G)}(Perf(\tilde{N}/G)) \\ Coh(St^d/G) \arrow[urr, "act", "\sim"'] & & \end{tikzcd}$$

we see act is LKEed from $(act \circ i)$.

Similarly, $\Phi : Coh(St^d/G) \longrightarrow D(\check{Fl})^{\check{I}}$ is LKEed from $(\Phi \circ i) \simeq \Phi_{perf}$.

Let $D(\check{Fl})^{\check{I}} \curvearrowright End(-)$ be denoted act_D. Recall we want $\Phi \circ \Psi' \simeq Id$,
i.e. $act \simeq act_D \circ \Phi$. By argument above, we get a natural transformation from
the following proposition:

<u>Prop</u>: $act \circ i \simeq act_D \circ \Phi_{perf}$. (This is proven using similar argument as fully faithfulness of Φ_{perf}).

Then to check the Nat is an iso, it suffices to do so on objects.
Unpacking definition we see that it amounts to check $\forall F \in Coh(St^d/G)$, $P \in Perf(\tilde{N}/G)$
we have: $\Phi_{IW}(F * P) \simeq \Phi(F) * \Phi_{IW}(P)$.

Noting that Nat between finite-amplitude functors is iso iff their restriction to Perf is,
it suffices to check that $\Phi_{IW}(- * P)$ and $\Phi(-) * \Phi_{IW}(P)$ have finite amplitude.
This follows from our knowledge about Φ and Φ_{IW}.