# NOTES ON [AB] (AFFINE FLAGS AND THE DUAL GROUP)

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## 1. Goals

Let $k$ be an algebraically closed field, over which the reductive group $G$ is defined. Let $e$ be an algebraically closed field of characteristic 0, over which the the Landlands dual group $\check{G}$ is defined. We shall consider sheaf theories according the choices of $k$ and $e$: for an algebro-geometric object $\mathcal{Y}$ defined over $k$:

- If $k = \overline{\mathbb{F}}_p, e = \overline{\mathbb{Q}}_l$, we can consider $\mathrm{Shv}^l(\mathcal{Y})$, which is the ind-completion of (non-cocomplete) DG-category[1] of $l$-adic constructible sheaves on $\mathcal{Y}$.
- If $k = \mathbb{C}, e = \mathbb{C}$, we can consider $\mathrm{DMod}(\mathcal{Y})$.
- If $k = \mathbb{C}$, we can consider $\mathrm{Shv}^{an}(\mathcal{Y})$, which is the ind-completion of (non-cocomplete) DG-category of constructible sheaves with coefficient $e$ for the analytic topology on $\mathcal{Y}^{an}$.

We shall just write Shv for any of the above cases.

The first goal of this talk is to construct an exact (commuting with finite colimits and limits in the $\infty$-categorical sense) monoidal functor

$$\Phi_{\mathrm{diag}} : \mathrm{Coh}(\check{\mathfrak{n}}/\check{B}) \to \mathrm{Shv}(\mathrm{Fl})^{I,lc},$$

where the $\mathrm{Shv}(\mathrm{Fl})^{I,lc}$ is the full subcategory of $\mathrm{Shv}(\mathrm{Fl})^I$ consisting of objects whose image in Shv(Fl) is compact (i.e. locally compact). It's known that the canonical involution on $\mathrm{Shv}(\mathrm{Fl})^I$ fixes this full subcategory. It's also known that the convolution preserves this full subcategory. The ind-completion of $\mathrm{Shv}(\mathrm{Fl})^{I,lc}$ is denoted by $\mathrm{Shv}(\mathrm{Fl})^I_{\mathrm{ren}}$, whose difference with $\mathrm{Shv}(\mathrm{Fl})^I$ is supported at cohomological degree of $-\infty$.

We warn that $\Phi_{\mathrm{diag}}$ is not t-exact.

Note that $\check{\mathfrak{n}}/\check{B} \simeq \tilde{\tilde{\mathcal{N}}}/\check{G}$, and we have a diagonal embedding $\tilde{\tilde{\mathcal{N}}}/\check{G} \hookrightarrow \mathrm{St}_{\check{G}}$. The above $\Phi_{\mathrm{diag}}$ will be the restriction of the desired equivalence

$$\Phi : \mathrm{Coh}(\mathrm{St}_{\check{G}}/\check{G}) \simeq \mathrm{Shv}(\mathrm{Fl})^{I,lc}$$

along the diagonal embedding $\tilde{\tilde{\mathcal{N}}} \hookrightarrow \mathrm{St}_{\check{G}}$. We warn that here $\mathrm{St}_{\check{G}}$ is the derived Steinberg variety.

The second goal of this talk is to describe an equivalence

$$\mathrm{Coh}(\check{\mathfrak{n}}/\check{B}) \simeq \mathrm{Shv}(\mathrm{Fl})^{(I_u^-,\chi),lc}$$

compatible with the monoidal action of $\mathrm{Coh}(\check{\mathfrak{n}}/\check{B})$ on both sides. Here $I^-$ is the opposite Iwahori, and $I_\mu^-$ is its unipotent radical. $\chi : I^- \to N^- \to \mathbb{G}_a$ is the generic character, and $\mathrm{Shv}(\mathrm{Fl})^{(I_u^-,\chi)}$ is the full subcategory of Shv(Fl) consisting of objects which are equivariant for $I_u^-$ against the character $\chi$. The $\mathrm{Coh}(\check{\mathfrak{n}}/\check{B})$-action on LHS is given by tensor product, while its action on RHS is induced by $\Phi_{\mathrm{diag}}$ and the convolution action.

$\mathrm{Shv}(\mathrm{Fl})^{(I_u^-,\chi),lc}$ is known as the Iwahori-Whittaker category. It's also called anti-spherical category because its de-categorify is the anti-spherical representation of the Iwahori-Hecke algebra.

We warn that the above equivalence is only right t-exact. In fact, the canonical t-structure on RHS corresponds to the *exotic* t-structure on LHS developed by R.B.

[1] I admit that I didn't check whether *all* the results in [AB] on triangulated categories can be generalized to $\infty$-categories. Therefore I refused to sign my real name.

## 2. Construction of $\Phi_{\text{diag}}$

2.1. **The plan.** Recall we have a closed embedding $\widetilde{\widetilde{\mathcal{N}}} \hookrightarrow \check{G}/\check{B} \times \check{\mathfrak{g}}$. Hence in order to construct $\Phi_{\text{diag}}$, it's enough to first construct an exact monoidal functor out of the category $\operatorname{Coh}(\check{G}\backslash(\check{G}/\check{B} \times \check{\mathfrak{g}}))$ and then provide certain vanishing data. This can be further separated into steps

- Step 1: Construct an exact monoidal functor out of $\operatorname{Coh}(\mathrm{pt}/\check{T})$;
- Step 2: Upgrade the above functor to an exact monoidal functor out of $\operatorname{Coh}(\mathrm{pt}/\check{B})$;
- Step 3: Construct an exact monoidal functor out of $\operatorname{Coh}(\check{G}\backslash\check{\mathfrak{g}})$;
- Step 4: Combining the above functors to an exact monoidal functor $\operatorname{Coh}(\check{G}\backslash(\check{G}/\check{B} \times \check{\mathfrak{g}}))$ and factors it through $\operatorname{Coh}(\check{G}\backslash\widetilde{\widetilde{\mathcal{N}}})$.

The construction in step 1 is given by the Wakimoto sheaves; that in step 2 uses the central sheaves and their filtrations by Wakimoto sheaves; that in step 3 uses the central sheaves and their monodromy; that in step 4 is provided by certain vanishing property of the monodromy on the graded piece of the central sheaves.

2.2. **Step 1: the Wakimoto sheaves.** The first step is to construct the functor

$$\operatorname{Coh}(\mathrm{pt}/\check{T}) \to \operatorname{Shv}(\mathrm{Fl})^{I,lc}.$$

For each coweight $\lambda \in \Lambda$ of $T$, we need to construct an object $J_\lambda$ in $\operatorname{Shv}(\mathrm{Fl})^{I,lc}$ and isomorphisms $J_\lambda \star J_\mu \simeq J_{\lambda+\mu}$ with higher compatibilities.

Recall as a set $I\backslash G(K)/I \simeq W^{\text{aff}}$, where $W^{\text{aff}}$ is the extended affine Weyl group (i.e. we have $0 \to \Lambda \to W^{\text{aff}} \to W \to 1$). For each $w \in W^{\text{aff}}$, consider the corresponding orbit $\mathrm{Fl}_w$, which is smooth of dimension $l(w)$, where $l(w)$ is the length of $w$. Define $j_{!,w}$ and $j_{*,w}$ respectively to be the !-extension and $*$-extension of the IC-sheaf on $\mathrm{Fl}_w$ to Fl, viewed as objects in $\operatorname{Shv}(\mathrm{Fl})^{I,lc}$.

**Exercise 2.2.1.** [2] *(1) $j_{w,*}, j_{w,!}$ is contained in $\operatorname{Shv}(\mathrm{Fl})^{I,lc,\heartsuit}$.*

*(2) $j_{w_1,!} \star j_{w_2,*}$ and $j_{w_1,*} \star j_{w_2,!}$ are contained in $\operatorname{Shv}(\mathrm{Fl})^{I,lc,\heartsuit}$.*

*(3) If $l(w_1 w_2) = l(w_1) l(w_2)$, then there are canonical isomorphisms $j_{w_1,*} \star j_{w_2,*} \simeq j_{w_1 w_2,*}$ and $j_{w_1,!} \star j_{w_2,!} \simeq j_{w_1 w_2,!}$ satisfy higher compatibilies.*

*(4) $j_{w,*} \star j_{w^{-1},!} \simeq \delta_e \simeq j_{w,!} \star j_{w^{-1},*}$.*

(Hints: For (1), one needs the fact that $\mathrm{Fl}_w \hookrightarrow \mathrm{Fl}$ is an *affine* locally closed embedding. For (2), one needs $\mathrm{Fl}_{w_1} \widetilde{\times} \mathrm{Fl} \to \mathrm{Fl}$ and $\mathrm{Fl} \widetilde{\times} \mathrm{Fl}_{w_2} \to \mathrm{Fl}$ are both affine. For (3), one needs $\mathrm{Fl}_{w_1} \widetilde{\times} \mathrm{Fl}_{w_2} \simeq \mathrm{Fl}_{w_1 w_2}$. Also, in an abelian category, higher compatibilities can be checked in finite time. For (4), do induction on $l(w)$. When $l(w) = 1$, do direct calculation on $\mathbb{P}^1 \widetilde{\times} \mathbb{P}^1$.)

It's known $l : W \to \mathbb{Z}$ is additive when restricted to the dominant coweight lattice $\Lambda^+$. Hence the following assignment is well-defined: $J_\lambda := j_{\lambda,*}$ when $\lambda \in \Lambda^+$; $J_\lambda := j_{\lambda,!}$ when $-\lambda \in \Lambda^+$; and $J_\lambda := j_{-\mu_1,!} \star j_{\mu_2,*}$ when $\lambda = \mu_2 - \mu_1$ with $\mu_1, \mu_2 \in \Lambda^+$ (such $\mu_1, \mu_2$ always exist).

By formal nonsense, there exists an unique monoidal exact and t-exact functor

$$\operatorname{Coh}(\mathrm{pt}/\check{T}) \to \operatorname{Shv}(\mathrm{Fl})^{I,lc},\ e_\lambda \mapsto J_\lambda,$$

where $e_\lambda$ is the 1-dimensional representation of $\check{T}$ with character $\lambda$.

2.3. **Step 2: the Drinfeld-Plucker formalism.** Now we want to upgrade the monoidal functor

$$\operatorname{Coh}(\mathrm{pt}/\check{T}) \to \operatorname{Shv}(\mathrm{Fl})^{I,lc},\ e_\lambda \mapsto J_\lambda. \tag{2.1}$$

to a monoidal functor

$$\operatorname{Coh}(\mathrm{pt}/\check{B}) \to \operatorname{Shv}(\mathrm{Fl})^{I,lc}.$$

Note that $\check{B} = \check{G}\backslash(\check{G}/\check{U})/\check{T}$. Hence it's enough to provide another monoidal functor $\operatorname{Coh}(\mathrm{pt}/\check{G}) \to \operatorname{Shv}(\mathrm{Fl})^{I,lc}$, equipped with certain compatibilities with (2.1). Thanks to D.G., we do have an exact and t-exact monoidal functor (the central sheaf construction)

$$\mathcal{Z} : \operatorname{Coh}(\mathrm{pt}/\check{G}) \to \operatorname{Shv}(\mathrm{Fl})^{I,lc},$$

[2] I believe exercises should never exist in any math writing. Therefore I refused to sign my real name.

and the classical study on Hecke algebras suggests that it is the pursured functor.

Note that $\mathcal{Z}$ can be upgraded to an $E_2$-functor from $\mathrm{Coh}(\mathrm{pt}/\check{G})$ to the Drinfeld center of $\mathrm{Shv}(\mathrm{Fl})^I_{\mathrm{ren}}$. Informally, this means we have functorial isomorphisms $\mathcal{Z}(V)\star\mathcal{F}\simeq\mathcal{F}\star\mathcal{Z}(V)$ satisfying higher compatibilities. This additional structure on $\mathcal{Z}$ allows one to define an exact and t-exact monoidal functor

$$\mathrm{Coh}(\check{G}\backslash\mathrm{pt}/\check{T})\to\mathrm{Shv}(\mathrm{Fl})^{I,lc},\ (V,\lambda)\mapsto\mathcal{Z}(V)\star J_\lambda. \tag{2.2}$$

Let $F:\mathrm{Coh}(\check{G}\backslash\mathrm{pt}/\check{T})\to\mathcal{C}$ be an exact monoidal functor. It remains to spell out the required (addition data of) compatibilities to upgrade it to an exact monoidal functor $\mathrm{Coh}(\mathrm{pt}/\check{B})\to\mathcal{C}$, which is just some formal nonsense, and then provide them for (2.2), which is some geometry representation theory. The formal nonsense is known as the Drinfeld-Plucker formalism, which we describe below.

In practice, it is easier to first extend $F$ to an exact monoidal functor $\mathrm{Coh}(\check{G}\backslash\overline{\check{G}/\check{U}}/\check{T})$, where $\overline{\check{G}/\check{U}}$ is the affine closure of $\check{G}/\check{U}$, and then verify that the result factors through $\mathrm{Coh}(\check{G}\backslash(\check{G}/\check{U})/\check{T})$. Note that such a factorization is a property rather than additional data.

**Exercise 2.3.1.** *(1) Let $V^\lambda$ be the highest weight module of $\check{G}$. Then as commutative algebra objects in $\mathrm{Rep}(\check{G}\times\check{T})$, we have*[3]

$$\mathcal{O}_{\overline{\check{G}/\check{U}}}\simeq\bigoplus_{\Lambda^+}V^\lambda\otimes e^{-\lambda},$$

*where the multiplication is given by*

$$(V^\lambda\otimes e^{-\lambda})\otimes(V^\mu\otimes e^{-\mu})\simeq(V^\lambda\otimes V^\mu)\otimes e^{-\lambda-\mu}\to V^{\lambda+\mu}\otimes e^{-\lambda-\mu}.$$

*(2) For $\lambda\in\Lambda^+$, write $\mathcal{V}^\lambda$ for the image of $V^\lambda$ under the pullback along $\check{G}\backslash\overline{\check{G}/\check{U}}/\check{T}\to\check{G}\backslash\mathrm{pt}$, and $\mathcal{L}^\lambda$ for the image of $e^\lambda$ under the pullback along $\check{G}\backslash\overline{\check{G}/\check{U}}/\check{T}\to\mathrm{pt}/\check{T}$. Check that the canonical morphisms*

$$\mathbf{b}^\lambda:\mathcal{V}^\lambda\simeq\bigoplus_{\mu\in\Lambda^+}(V^\lambda\otimes V^\mu)\otimes e^{-\mu}\to\bigoplus_{\mu\in\Lambda^+}(V^{\lambda+\mu})\otimes e^{-\mu}\to\bigoplus_{\mu\in\Lambda^+}V^\mu\otimes e^{-\mu+\lambda}\simeq\mathcal{L}^\lambda$$

*satisfying the Plucker condition*

$$\begin{array}{ccc}\mathcal{V}^{\lambda+\mu} & \longrightarrow & \mathcal{V}^\lambda\otimes\mathcal{V}^\mu\\ \downarrow{\scriptstyle \mathbf{b}^{\lambda+\mu}} & & \downarrow{\scriptstyle \mathbf{b}^\lambda\otimes\mathbf{b}^\mu}\\ \mathcal{L}^{\lambda+\mu} & \xrightarrow{\simeq} & \mathcal{L}^\lambda\otimes\mathcal{L}^\mu\end{array}$$

*and higher compatibilities.*

*(3) Let $d(\lambda)=\dim(V^\lambda)$. Show that the Koszul complex associated to $\mathbf{b}^\lambda$:*

$$0\to\wedge^{d(\lambda)}\mathcal{V}^\lambda\to\wedge^{d(\lambda)-1}\mathcal{V}^\lambda\otimes\mathcal{L}^\lambda\to\cdots\to\mathcal{V}^\lambda\otimes\mathcal{L}^{(d(\lambda)-1)\lambda}\to\mathcal{L}^{d(\lambda)\lambda}\to 0$$

*vanishes when restricted to the open $\check{G}\backslash(\check{G}/\check{U})/\check{T}$.*

By the exercise, in order to extend $F$ to an exact monoidal functor $\mathrm{Coh}(\check{G}\backslash\overline{\check{G}/\check{U}}/\check{T})\to\mathcal{C}$, we at least need to construct morphisms $\mathbf{b}^\lambda:F(V^\lambda)\to F(e^\lambda)$ satisfying Plucker conditions and certain higher compatibilities. If we want this extension to factor through $\mathrm{Coh}(\mathrm{pt}/B)$, we at least need to check that the Koszul complexes associated to the above $\mathbf{b}^\lambda$ vanish.

It turns out that the above necessary data and conditions are also sufficient[4]. It remains to provide them for our functor (2.2).

**Exercise 2.3.2.** *For $\lambda\in\Lambda^+$, write $\mathcal{Z}_\lambda$ for $\mathcal{Z}(V^\lambda)$.*

*(1) As objects in $\mathrm{Shv}(\mathrm{Fl}_\lambda)^{I,lc}$, we have a canonical isomorphism*

$$j_\lambda^*(\mathcal{Z}_\lambda)\simeq\mathrm{IC}_{\mathrm{Fl}_\lambda}.$$

*(2) By (1), we have a canonical isomorphism between spaces*[5]

$$\mathrm{Maps}_{\mathrm{Shv}(\mathrm{Fl})^{I,lc}}(\mathcal{Z}_\lambda,J_\lambda)\simeq e.$$

[3] The negative sign comes from the convention that $T$ acts leftly on $G/U$ via $t\cdot gU:=gt^{-1}U$.

[4] [AB] proved it for triangulated categories. [D.G., semi-infinite IC sheaf] generalized it to $\infty$-categories.

[5] Here $\mathrm{Maps}(-,-)$ is the space of morphisms, rather than its enrichment in Vect.

*Let* $\mathbf{b}^\lambda : \mathcal{Z}_\lambda \to J_\lambda$ *be the morphism corresponding to the canonical generator of* $e$*. Verify the Plucker conditions and higher compatibilities for* $\mathbf{b}^\lambda$*.*

(Hint: for (1), nearby-cycle commuting with proper push-forward implies that $\dim \mathrm{Supp}(\mathcal{Z}_\lambda) \leq \dim(\mathrm{Gr}_\lambda) = \dim(\mathrm{Fl}_\lambda)$. Combinatorics show that $\mathrm{Fl}_\lambda$ is the only $I$-orbit in the preimage of $\mathrm{Gr}_\lambda$ such that it dominates $\mathrm{Gr}_\lambda$ and has dimension equal to $\dim(\mathrm{Gr}_\lambda)$. Then one wins by the base-change isomorphisms together with the fact that the push-forward of $\mathcal{Z}_\lambda$ is the Satake sheaf on Gr.

For (2), the Plucker condition is a formal consequence of the fact that the push-forward of $\mathcal{Z}_\lambda$ is the Satake sheaf on Gr. For the higher compatibilities, fortunately all the calculations live in $\mathrm{Shv}(\mathrm{Fl})^{I,lc,\heartsuit}$, hence they can be checked in a finite time.)

It remains to show that the Koszul complex associated to the above $\mathbf{b}^\lambda$ vanishes on $\mathrm{pt}/B$. The strategy to do this is as follows. Let fix a total ordering on $\Lambda$ that extends the usual partial ordering given by positive roots. Suppose we can upgrade the functor (2.2) to a functor $\mathrm{Coh}(\mathrm{pt}/B) \to \mathrm{Shv}(\mathrm{Fl})^{I,lc}$, then it's easy to see $\mathcal{Z}_\lambda$ should be equipped with a unique filtration by $\Lambda$ whose graded piece is $J_\mu \otimes V^\lambda(\mu)$, where $V^\lambda(\mu)$ is the $\mu$-weight subspace in $V^\lambda$. Moreover, these filtrations should be compatible with the convolutions in the obvious sense.

Conversely, suppose we already have these filtrations such that the canonical map $\mathcal{Z}_\lambda \to \mathrm{gr}^\lambda(\mathcal{Z}_\lambda) \simeq J_\lambda$ is $\mathbf{b}^\lambda$, then it's easy to verify the desired vanishing property. Hence it remains to construct such filtrations, whose details are actually not covered in this talk. However, let's point out:

- The existence of *some* filtrations on $\mathcal{Z}_\lambda$ by *generalized* Wakimoto sheaves ($J_\lambda \star j_{w,*}$ for $\lambda \in \Lambda, w \in W$) is a formal consequence of the fact that convolution with $\mathcal{Z}_\lambda$ (on both sides) is t-exact.
- The claim that only the Wakimoto sheaves $J_\lambda$ appear in the above filtration is a formal consequence of the fact $\mathcal{Z}_\lambda$ is central.

Let's also mention that the resulting functor $\mathrm{Coh}(\mathrm{pt}/\check{B}) \to \mathrm{Shv}(\mathrm{Fl})^{I,lc}$ is t-exact.

2.4. **Step 3: the monodromy.** We want to upgrade the exact and t-exact monoidal functor

$$\mathcal{Z} : \mathrm{Coh}(\mathrm{pt}/\check{G}) \to \mathrm{Shv}(\mathrm{Fl})^{I,lc}$$

to an exact monoidal functor out of $\mathrm{Coh}(\check{\mathfrak{g}}/\check{G})$.

**Exercise 2.4.1.** *Construct a canonical symmetric monoidal endomorphism* $\mathbf{N}$ *on the pullback functor* $\mathrm{Coh}(\mathrm{pt}/\check{G}) \to \mathrm{Coh}(\check{\mathfrak{g}}/\check{G})$ *such that on the level of objects, the corresponding endomorphism* $\mathbf{N}_V \in \mathrm{End}_{\mathcal{O}_{\check{\mathfrak{g}}}\text{-mod}(\mathrm{Rep}(\check{G}))}(\mathcal{O}_{\check{\mathfrak{g}}} \otimes V) \simeq \mathrm{Maps}_{\mathrm{Rep}(\check{G})}(V, \mathcal{O}_{\check{\mathfrak{g}}} \otimes V)$ *corresponds to the usual action of the Lie algebra* $\check{\mathfrak{g}}$ *on* $V \in \mathrm{Rep}(G)$*.*

The above exercise suggests that we should at least construct a monoidal endomorphism $\mathbf{N} : \mathcal{Z} \to \mathcal{Z}$ satisfying certain higher compatibilities with the commutativity constraints for the central sheaves. It turns out the above data is also sufficient.[6]

Recall $\mathcal{Z}$ is constructed via nearby-cycles. It's known that the monodromy on these nearby-cycles is unipotent. In particular, the logarithm of monodromy is well-defined, and is an endomorphism $\mathbf{N}$ on the functor $\mathcal{Z}$.

**Exercise 2.4.2.** *Using the Kunneth equivalences on nearby-cycles to show that* $\mathbf{N}$ *can be upgraded to a monoidal endomorphism satisfying all the desired properties.*

Therefore we obtained the desired exact monoidal functor

$$\mathrm{Coh}(\check{\mathfrak{g}}/\check{G}) \to \mathrm{Shv}(\mathrm{Fl})^{I,lc}.$$

[6] We actually don't know how to show this for $\infty$-categories. Note that although one starts from a t-exact functor $\mathcal{Z}$, the resulting extension will not be t-exact.

2.5. **Step 4: the vanishing data.** Note that we have a Cartesian square

$$\begin{CD} \check{G}\backslash(\check{G}/\check{U} \times \check{\mathfrak{g}})/\check{T} @>>> \check{G}\backslash(\check{G}/\check{U})/\check{T} \\ @VVV @VVV \\ \check{G}\backslash\mathfrak{g} @>>> \check{G}\backslash\mathrm{pt}. \end{CD}$$

Some formal nonsense allows us to glue step 2 and step 3 to an exact monoidal functor

$$\mathrm{Coh}(\check{G}\backslash(\check{G}/\check{U} \times \check{\mathfrak{g}})/\check{T}) \to \mathrm{Shv}(\mathrm{Fl})^{I,lc}.$$

**Exercise 2.5.1.** *(1) Giving a factorization of the above functor through* $\mathrm{Coh}(\check{G}\backslash\tilde{\tilde{\mathcal{N}}})$ *is equivalent to showing that* $\mathbf{b}_\lambda \circ \mathbf{N}_{\mathcal{Z}_\lambda} \simeq 0$.

*(2) Check* $\mathbf{b}_\lambda \circ \mathbf{N}_{\mathcal{Z}_\lambda} \simeq 0$.

(Hint: (1) is more or less by definition. (2) reflects the fact that $\mathbf{N}_{\mathcal{Z}_\lambda}$ is nilpotent.)

We finished the construction of $\Phi_{\mathrm{diag}}$. Let's explain why it is right t-exact. Indeed, $\mathrm{Coh}(\check{G}\backslash\tilde{\tilde{\mathcal{N}}})$ is the derived category of its heart, and every object in $\mathrm{Coh}(\check{G}\backslash\tilde{\tilde{\mathcal{N}}})^\heartsuit$ has a resolution by objects obtained by pullback from $\check{G}\backslash\mathrm{pt}$. Then we win because the images of $\Phi_{\mathrm{diag}}$ to these objects are contained in the heart.

## 3. The Iwahori-Whittaker category