LANGLANDS SUPPORT GROUP : SPECTRAL GLUING

0.1. What do we want?

0.1.1. We will devote this semester to studying how the category $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_G)$ interacts with parabolic subgroups of G. The ultimate goal is to obtain a more concrete understanding of $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_G)$ and why it should appear in geometric Langlands.

0.1.2. We will study two theorems (referred to as (a) and (b)) that serve our purpose:

(a) For each parabolic P with Levi quotient M, there is a spectral Eisenstein series functor:

$$\mathrm{Eis}^P_{\mathrm{Spec}} : \mathrm{QCoh}(\mathrm{LocSys}_M) \to \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_G).$$

The first theorem asserts that the target is generated by the images of $\mathrm{QCoh}(\mathrm{LocSys}_M)$, when P ranges through all parabolics (including $P = G$). It shows that $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_G)$ is *large enough* to contain all Eisenstein series coming from the Levi subgroups. Theorem (a) is proved in §13.3 of:

[AG1] http://www.math.harvard.edu/~gaitsgde/GL/singsupp.pdf.

(b) We let $\mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G}$ denote the category of quasi-coherent sheaves on LocSys_P with a connection along the fiber of $\mathfrak{p} : \mathrm{LocSys}_P \to \mathrm{LocSys}_G$. There is a restriction functor:

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_G) \to \mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G}$$

When P varies, we can form a "glued category" $\underset{P\in\mathrm{Par}(G)}{\mathrm{Glue}}(\mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G})$, and the second theorem asserts that the "combined" restriction functor:

$$\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_G) \to \underset{P\in\mathrm{Par}(G)}{\mathrm{Glue}}(\mathrm{QCoh}(\mathrm{LocSys}_P)_{\mathrm{conn}\,/\,\mathrm{LocSys}_G})$$

is fully faithful. As the latter category embeds into $\mathrm{Whit}^{\mathrm{ext}}(\check{G}, \check{G})$, theorem (b) shows that $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LocSys}_G)$ is *small enough* to be contained in (Langlands dual) extended Whittaker coefficients. It is proved as the main theorem of:

[AG2] http://www.math.harvard.edu/~gaitsgde/GL/Gluing.pdf.

0.2. How do we do it?

0.2.1. We will split the semester into 5 parts, each with its own focus. In what follows, week 1 refers to September 3–9, and week n is defined inductively.

0.2.2. *Week 2–4.* Overview. Theory of ind-coherent sheaves. Quasi-smoothness. Singular support.

0.2.3. *Week 5–6.* The stack LocSys_G and why it is quasi-smooth. The global nilpotent cone. Spectral Eisenstein series. Proof of theorem (a).

0.2.4. *Week 7–9.* Localizing the category of singularities $\overset{\circ}{\mathrm{IndCoh}}(Y)$ on $\mathbb{P}\,\mathrm{Sing}(Y)$ (Theorem 1.4.2 of [AG2]). Relative crystals.

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0.2.5. *Week 10–11.* Formulate theorem (b) and give the sufficient condition for spectral gluing in terms of $\mathcal{D}$-modules on the scheme of singularities (Theorem 4.4.5 of [AG2]).

0.2.6. *Week 12–14.* Prove theorem (b).