The category of singularity as a crystal

1\. · Z affine, almost of finite type, quasi-smooth

$$\overset{\circ}{\mathrm{IndCoh}}(Z) := \mathrm{IndCoh}(Z)/\mathrm{QCoh}(Z)$$

Ex1: $\Xi_Z : \mathrm{QCoh}(Z) \rightleftarrows \mathrm{IndCoh}(Z) : \Psi_Z$

$$\rightsquigarrow \ker \Psi_Z \simeq \overset{\circ}{\mathrm{IndCoh}}(Z) \simeq \mathrm{QCoh}(Z)^{\perp}.$$

(leave to end) **Ex2:** $\overset{\circ}{\mathrm{IndCoh}}(Z)$ inherits a $\mathrm{QCoh}(Z)$ action.

Recall: Z ~~quasi~~ $\rightsquigarrow$ $\mathrm{Sing}(Z)$ ~~$\curvearrowleft \mathbb{G}_m$~~ classical scheme

$$\begin{array}{ccc} & & \mathrm{Sing}(Z) \curvearrowleft \mathbb{G}_m \\ & \overset{\{0\}}{\nearrow} & \downarrow \\ {}^{cl}Z & & Z \end{array}$$

Def: $\mathbb{P}\mathrm{Sing}(Z) = (\mathrm{Sing}(Z) - \{0\})/\mathbb{G}_m$

Recall: Conical closed $N \subset \mathrm{Sing}(Z)$

$$\rightsquigarrow \mathrm{IndCoh}_N(Z) := \{\mathcal{F} \mid \mathrm{SingSupp}(\mathcal{F}) \subset N\}$$
$$\cap$$
$$\mathrm{IndCoh}(Z)$$

Recall: $\mathrm{IndCoh}_{\{0\}}(Z) = \mathrm{QCoh}(Z)$.

Def-Ex: For $\mathcal{F} \in \overset{\circ}{\mathrm{IndCoh}}(Z)$, choose $\mathcal{F}^{\#} \in \mathrm{IndCoh}(Z)$

$$\mathbb{P}\mathrm{SingSupp}(\mathcal{F}) := (\mathrm{SingSupp}(\mathcal{F}^{\#}) - \{0\})/\mathbb{G}_m$$

$$\subset \mathbb{P}\mathrm{Sing}(Z)$$

is independent of choice of $\mathcal{F}^{\#}$.

Def: For closed $N \subset \mathbb{P}\mathrm{Sing}(Z)$

$$\overset{\circ}{\mathrm{IndCoh}}(Z) := \{\mathcal{F} \mid \mathbb{P}\mathrm{SingSupp}(\mathcal{F}) \subset N\}.$$

Main Th'm:

There exists a canonical defined

$$\mathring{\mathrm{Ind}}\mathrm{Coh}(\mathcal{Z})^{\sim} \in \mathrm{ShvCat}(\mathbb{P}\mathrm{Sing}(\mathcal{Z})_{dR})$$

w/

$$\Gamma(\mathbb{P}\mathrm{Sing}(\mathcal{Z})_{dR}, \mathring{\mathrm{Ind}}\mathrm{Coh}(\mathcal{Z})^{\sim}) \simeq \mathring{\mathrm{Ind}}\mathrm{Coh}(\mathcal{Z})$$

s.t. For closed $\mathcal{N} \subset \mathbb{P}\mathrm{Sing}(\mathcal{Z})$

$$\begin{array}{ccc} \mathring{\mathrm{Ind}}\mathrm{Coh}_{\mathcal{N}}(\mathcal{Z}) & \hookrightarrow & \mathring{\mathrm{Ind}}\mathrm{Coh}(\mathcal{Z}) \\ \wr\!\! \mid & & \wr\!\! \mid \\ \Gamma(\mathcal{N}_{dR}, \mathring{\mathrm{Ind}}\mathrm{Coh}(\mathcal{Z})^{\sim}) & \overset{\text{(see next prop)}}{\dashrightarrow} & \Gamma(\mathbb{P}\mathrm{Sing}(\mathcal{Z})_{dR}, \mathring{\mathrm{Ind}}\mathrm{Coh}(\mathcal{Z})^{\sim}) \end{array}$$

• Prop: For closed embedding of laft prestacks $i: \mathcal{Z} \to \mathcal{Y}$

$$\mathcal{Z}_{dR} \overset{i_{dR}}{\hookrightarrow} \mathcal{Y}_{dR} \overset{j_{dR}}{\longleftarrow} \mathring{\mathcal{Y}}_{dR}.$$

a) $(i_{dR})_*^{\mathrm{ShvCat}}$ is fully faithful w/ essential image given by kernel of $(j_{dR})^{*,\mathrm{ShvCat}}$

b). For $\mathcal{C} \in \mathrm{ShvCat}(\mathcal{Y}_{dR})$

$$\begin{array}{ccc} \Gamma(\mathcal{Y}_{dR}, \mathcal{C}) & \longrightarrow & \Gamma(\mathcal{Z}_{dR}, \mathcal{C}) \\ \uparrow & \nearrow_{\sim} & \\ \ker(\Gamma(\mathcal{Y}_{dR}, \mathcal{C}) \to \Gamma(\mathring{\mathcal{Y}}_{dR}, \mathcal{C})) & & \end{array}$$

Proof:

$$\begin{tikzcd} Z_{dR} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Z_{dR} \arrow[d] \\ Z_{dR} \arrow[r] & Y_{dR} \end{tikzcd} \quad + \text{ base change } \Rightarrow (i_{dR})_*^{\mathrm{ShvCat}} \text{ fully faithful}$$

- $$\begin{tikzcd} S^{\wedge}_{S_0} \arrow[r] \arrow[d] & S \arrow[d] & S - S_0 = \mathring{S} \arrow[l] \arrow[d] \\ Z_{dR} \arrow[r, hook] & Y_{dR} & \mathring{Y}_{dR} \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} S_0 \arrow[r] \arrow[d] & S^{red} \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd}$$

Lemma: For S, S_0 as above

$$S^{\wedge}_{S_0} \xrightarrow{\hat{i}} S \xleftarrow{j} \mathring{S}$$

a) $\hat{i}_*^{\mathrm{ShvCat}}$ is fully faithful w/ essential image given by $\ker(j^{*,\mathrm{ShvCat}})$.

b) $\mathcal{C} \in \mathrm{ShvCat}(S)$

$$\begin{tikzcd} \Gamma(S, \mathcal{C}) \arrow[r] & \Gamma(\hat{S}_{S_0}, \mathcal{C}) \\ \ker(\Gamma(S, \mathcal{C}) \to \Gamma(\mathring{S}, \mathcal{C})) \arrow[u] \arrow[ur, "\sim"'] & \end{tikzcd}$$

Proof of Lemma: S is 1-affine, $S^{\wedge}_{S_0}$ is 1-affine

a) $\Leftrightarrow$ For $\mathbf{C} \in \mathrm{QCoh}(S)\text{-mod}$, if $\mathbf{C} \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(\mathring{S}) = 0$

then $\mathrm{QCoh}(S) \curvearrowright \mathbf{C}$
$$\begin{tikzcd} \mathrm{QCoh}(S) \arrow[d] \\ \mathrm{QCoh}(\hat{S}_{S_0}) \end{tikzcd}$$

b) $\Leftrightarrow$
$$\begin{tikzcd} \mathbf{C} \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(\hat{S}_{S_0}) & \mathbf{C} \arrow[l] \\ \ker(\mathbf{C} \to \mathbf{C} \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(\mathring{S})) \arrow[u, "\wr"] \arrow[ur, "\simeq"'] & \end{tikzcd}$$

$\square$

Motivation for Main Thm. (and its construction)

Temporarily allow Z to be a stack.

$Z = pt \times_V pt$, $V = T_{pt}(U)$, U is parallelized.

Recall: There exists an action $QCoh(V^*/\mathbb{G}_m)$ on $IndCoh(Z)$ s.t.

$$IndCoh_N(Z) = IndCoh(Z) \otimes_{QCoh(V^*/\mathbb{G}_m)} QCoh(V^*/\mathbb{G}_m)_{N/\mathbb{G}_m}$$

(by Koszul duality)

Expect to make $IndCoh(Z)$ as Γ of a sheaf on $(V^*/\mathbb{G}_m)_{dR}$.

Trouble: $(V^*/\mathbb{G}_m)_{dR}$ is not 1-affine.

Solution: $((V^* - \{0\})/\mathbb{G}_m)_{dR} = \mathbb{P}V^*_{dR}$ is 1-affine.

$QCoh(V^*/\mathbb{G}_m) \curvearrowright IndCoh(Z)$

$QCoh(\mathbb{P}V^*) \curvearrowright \overset{\circ}{IndCoh}(Z)$

$\uparrow$

$Dmod(\mathbb{P}V^*)$

This will ~~be our~~ give $IndCoh(Z)^\sim$.

(Explicitly:

$IndCoh(Z) \simeq Sym(V[-2])$-mod

$QCoh(V^*/\mathbb{G}_m) = (Sym\, V[-2])$-mod$^{\mathbb{G}_m}$

("Magical grading shift"))

Also works for:

$$\begin{array}{ccc} Z & \longrightarrow & \mathcal{U} \\ \downarrow & & \downarrow \\ pt & \longrightarrow & V \end{array} \qquad \mathrm{Sing}(Z) \hookrightarrow V^* \times \mathcal{U}$$

<u>Recall</u>: $\mathrm{QCoh}(V^*/\mathbb{G}_m \times \mathcal{U}) \curvearrowright \mathrm{IndCoh}(Z)$

$$\mathrm{IndCoh}_{\mathcal{N}}(Z) \simeq \mathrm{IndCoh}(Z) \underset{\mathrm{QCoh}(V^*/\mathbb{G}_m \times \mathcal{U})}{\otimes} \mathrm{QCoh}(V^*/\mathbb{G}_m \times \mathcal{U})_{\mathcal{N}/\mathbb{G}_m}$$

$$\rightsquigarrow \quad \mathrm{QCoh}(\mathbb{P}V^* \times \mathcal{U}) \curvearrowright \mathrm{Ind}\overset{\circ}{\mathrm{Coh}}(Z)$$

$$\begin{array}{c} \Uparrow \\ \cancel{\mathrm{IndCoh}}\ \mathrm{Dmod}(\mathbb{P}V^* \times \mathcal{U}) \\ \downarrow \\ \mathrm{Dmod}(\mathbb{P}\,\mathrm{Sing}(Z)) \end{array}$$

$\dashrightarrow$ Using the proposition checking the restriction to $(\mathbb{P}V^* \times \mathcal{U} - \mathbb{P}\,\mathrm{Sing}(Z))_{dR}$ is zero

<u>The Clean construction</u>:

<u>Setting</u>: $\mathbb{E}_2$-algebra $\mathcal{A}$, even graded. Even finitely generated algebra A w/

$$A \longrightarrow H^0(\mathcal{A})$$

$$\mathcal{A} \curvearrowright \mathbb{C}$$

$\mathbb{C} = \mathrm{IndCoh}(Z)$
$\mathcal{A} = HC(Z)$
$A = \Gamma(\mathrm{Sing}(Z), \mathcal{O})$

<u>Recall</u>: We have the theory of ~~singular~~ support on $\mathrm{Spec}(A)$.

$$\overset{\circ}{\mathbb{C}} := \mathbb{C} / \mathbb{C}_{\{0\}}, \quad \{0\} = \mathrm{Spec}(A^0) \hookrightarrow \mathrm{Spec}\,A.$$

We can define $\overset{\circ}{\mathbb{C}}_{\mathcal{N}}$ similarly.

Construction:

We construct $\mathcal{C} \overset{=}{} \mathcal{C}_A \in \mathrm{ShvCat}(\mathrm{Proj}(A)_{dR})$:

For $S \to \mathrm{proj}(A)_{dR}$

$(S^{red} \xrightarrow{f} \mathrm{proj}(A))$

$\Gamma(S, \mathcal{C})$ $\left(\underset{\text{want}}{=} \mathrm{QCoh}(S) \underset{\mathrm{Dmod}(\mathrm{proj}(A))}{\otimes} \mathring{\mathbb{C}} \right)$

$\cap$

$(\mathrm{QCoh}(S) \otimes \mathring{\mathbb{I}})_{\Gamma}$

is defined as $(\mathrm{QCoh}(S) \otimes \mathring{\mathbb{I}})_{\Gamma_f}$

$\left(\begin{array}{c} \mathcal{O}_S \otimes A \to \mathcal{O}_S \otimes H^0(A) \\ \mathcal{O}_S \otimes A \curvearrowright \mathrm{QCoh}(S) \otimes \mathbb{I} \end{array} \right)$

Ex: $S = pt$, ~~$\Gamma(pt$~~, $\mathcal{C}|_{\{f\}} = \mathring{\mathbb{I}}_{\{f\}}$.

Ex: Functorial wrt S.

Ex: $\mathring{\mathbb{I}} \xrightarrow{\mathcal{O}_S \otimes -} \mathrm{QCoh}(S) \otimes \mathring{\mathbb{C}} \xrightarrow{\text{co-localization}} (\mathrm{QCoh}(S) \otimes \mathring{\mathbb{C}})_{\Gamma_f}$.

functorial wrt S.

$\rightsquigarrow \mathring{\mathbb{I}} \xrightarrow{\partial_A} \Gamma(\mathrm{Proj}(A)_{dR}, \mathcal{C})$

Ex: Using prop to show

$$\begin{tikzcd} \mathring{\mathbb{I}}_N \arrow[r, dashed, "\partial_{A,V}"] \arrow[d, hook] & \Gamma(N_{dR}, \mathcal{C}) \arrow[d, hook] \\ \mathring{\mathbb{C}} \arrow[r] & \Gamma(\mathrm{proj}(A)_{dR}, \mathcal{C}) \end{tikzcd}$$

Proof of Main Th'm

~~Ex: Using our direct computation for $\mathcal{Z} = pt \times_V \mathcal{U}$ to show both maps are isomorphism.~~

It remains to show

$$\mathring{\mathbb{C}}_N \xrightarrow[\sim]{\partial_{A,N}} \Gamma(N_{dR}, \mathcal{C}_A)$$

for our (Indcoh) case.

- Local for $\mathcal{Z}$ $\rightsquigarrow$ assume $\mathcal{Z} = pt \times_V \mathcal{U}$.

Recall: $\mathcal{B} := \Gamma(\mathcal{U}, \mathcal{O}_\mathcal{U}) \otimes \mathrm{Sym}(V[-2])$

$$\downarrow$$

$\mathcal{A} = HC(\mathcal{Z})$

$$\begin{array}{c} \mathrm{Proj}(B) \\ \downarrow p \\ \mathrm{Proj}(A)_{dR} \xrightarrow{g_{dR}} \mathrm{Proj}(B)_{dR} \end{array}$$

$B := H^\bullet(\mathcal{B})$, $B \to A$

$\mathrm{Spec}\, A \to \mathrm{Spec}\, B$ ~~not~~ closed embedding

Claim 1: $\mathcal{C}_B \simeq p_*^{\mathrm{ShvCat}}(\mathbb{1}\mathrm{Loc}_{\mathrm{Proj}(B)}(\mathring{\mathbb{C}}))$

s.t. ∂_B is given by

$$\mathring{\mathbb{C}} \underset{(1\text{-affineness})}{\xrightarrow{\sim}} \Gamma(\mathrm{Proj}(B), \mathbb{1}\mathrm{Loc}(\mathring{\mathbb{C}})) \simeq \Gamma(\mathrm{Proj}(B)_{dR}, p_*^{\mathrm{ShvCat}}(\mathbb{1}\mathrm{Loc}(\mathring{\mathbb{C}})))$$

$$\wr\wr$$

$$\Gamma(\mathrm{Proj}(B)_{dR}, \mathcal{C}_B)$$

Claim 2: 1) • $(g_{dR})^{*,\mathrm{ShvCat}}(\mathcal{C}_B) \simeq \mathcal{C}_A$

2) • By adjunction

$$\mathcal{C}_B \longrightarrow (g_{dR})_*^{\mathrm{ShvCat}}(\mathcal{C}_A)$$

which is an iso.

Claim 2 $\Rightarrow$

$$\Gamma(\mathrm{Proj}(B)_{dR}, \mathcal{C}_B) \simeq \Gamma(\mathrm{Proj}(A)_{dR}, \mathcal{C}_A)$$

+ Claim 1 $\Rightarrow$ ∂_A is Bo.

- For $N \subset \mathrm{Proj}(A)$, assume $N = \mathbb{P}(\mathcal{N}^{\#})$. Replace $\mathbb{C}$ by $\mathbb{C}_{\mathcal{N}^{\#}}$ and repete the above argument $\rightsquigarrow$ $\partial_{A,N}$ is Bo.

· Proof of claim 1:

For $S^{red} \longrightarrow \mathrm{Proj}(B)_{dR}$ $\qquad S^{red} \xrightarrow{f} \mathrm{Proj}(B)$

$$\Gamma(S, p_*^{ShvCat}(\mathbb{L}oc_{\mathrm{Proj}(B)}(\mathring{\mathbb{C}})))$$

$$\|$$

$$\Gamma((S \times \mathrm{Proj}(B))^{\wedge}_{\Gamma_f}, \mathbb{L}oc_{\mathrm{Proj}(B)}(\mathring{\mathbb{C}}))$$

$$\|$$

$$\mathring{\mathbb{C}} \underset{QCoh(\mathrm{Proj}(B))}{\otimes} QCoh(S \times \mathrm{Proj}(B))_{\Gamma_f}$$

$$\|$$

$$(\mathring{\mathbb{C}} \otimes QCoh(S))_{\Gamma_f}$$

$\square$

° Proof of Claim 2.

1) is Trivial

2) For $\mathcal{F} \in \mathring{\mathbb{C}}$, $\mathbb{P}\mathrm{supp}_B(\sigma_{\mathcal{F}}) \subset \mathrm{Proj}(A) \cap \mathrm{Proj}(B)$

so its restriction to $\mathrm{Proj}(B) - \mathrm{Proj}(A)$ is 0.

Now by the [illegible] Sp.

How to update it to stack?

$\mathcal{Z}$ is quasi-smooth algebraic stack w/ affine diagonals

Thm: $\mathcal{Z}$ is 1-affine.

Rmk: $\mathcal{Z}_{dR}$ is not 1-affine!

Recall: ShvCat: satisfying smooth descent:
In order to construct $\mathcal{C}$ on $\mathcal{Y}$, enough to do it on each $Y \xrightarrow{\text{smooth}} \mathcal{Y}$ compatibly.

Trouble: ~~For~~ Although for $Z \to \mathcal{Z}$ smooth,

$$\mathbb{P}\mathrm{Sing}(\mathcal{Z})_{dR} \underset{\mathcal{Z}_{dR}}{\times} Z_{dR} \simeq \mathbb{P}\mathrm{Sing}(Z)_{dR}$$

but $Z_{dR} \to \mathcal{Z}_{dR}$ is not smooth!

Solution: Update the main theorem to a shvcat on

$$\mathbb{P}\mathrm{Sing}(\mathcal{Z})_{dR} \underset{\mathcal{Z}_{dR}}{\times} \mathcal{Z}$$

then

$$\left(\mathbb{P}\mathrm{Sing}(\mathcal{Z})_{dR} \underset{\mathcal{Z}_{dR}}{\times} \mathcal{Z}\right) \underset{\mathcal{Z}}{\times} Z \simeq \mathbb{P}\mathrm{Sing}(Z)_{dR} \underset{Z_{dR}}{\times} Z$$

would glue together.

~~Construction~~:

~~Updated Main Theorem~~

- Lemma: For Z scheme,

$$(\mathbb{P}\mathrm{Sing}(Z))_{dR} \underset{Z_{dR}}{\times} Z \quad \text{is } 1\text{-affine}$$

Proof: it's a closed subprestack of

$$(\mathbb{P}^N)_{dR} \times Z$$

- It remains to update the action given by the main theorem:

$$\begin{array}{ccc} \mathrm{QCoh}(Z_{dR}) \to \mathrm{QCoh}(\mathrm{Sing}(Z)_{dR}) & \curvearrowright & \mathrm{IndCoh}^{0}(Z) \\ \downarrow & & \\ \mathrm{QCoh}(\mathrm{Sing}(Z)_{dR} \underset{Z_{dR}}{\times} Z) & & \end{array}$$

$\Longleftrightarrow$ Give an action $\mathrm{QCoh}(Z) \curvearrowright \mathrm{IndCoh}^{0}(Z)$ s.t. it's restriction to $\mathrm{Dmod}(Z)$ is ~~the~~ compatible with that comes from

$$\mathrm{D}(\mathrm{Sing}(Z)_{dR})$$

- Fine! (Can be checked by own computation!)