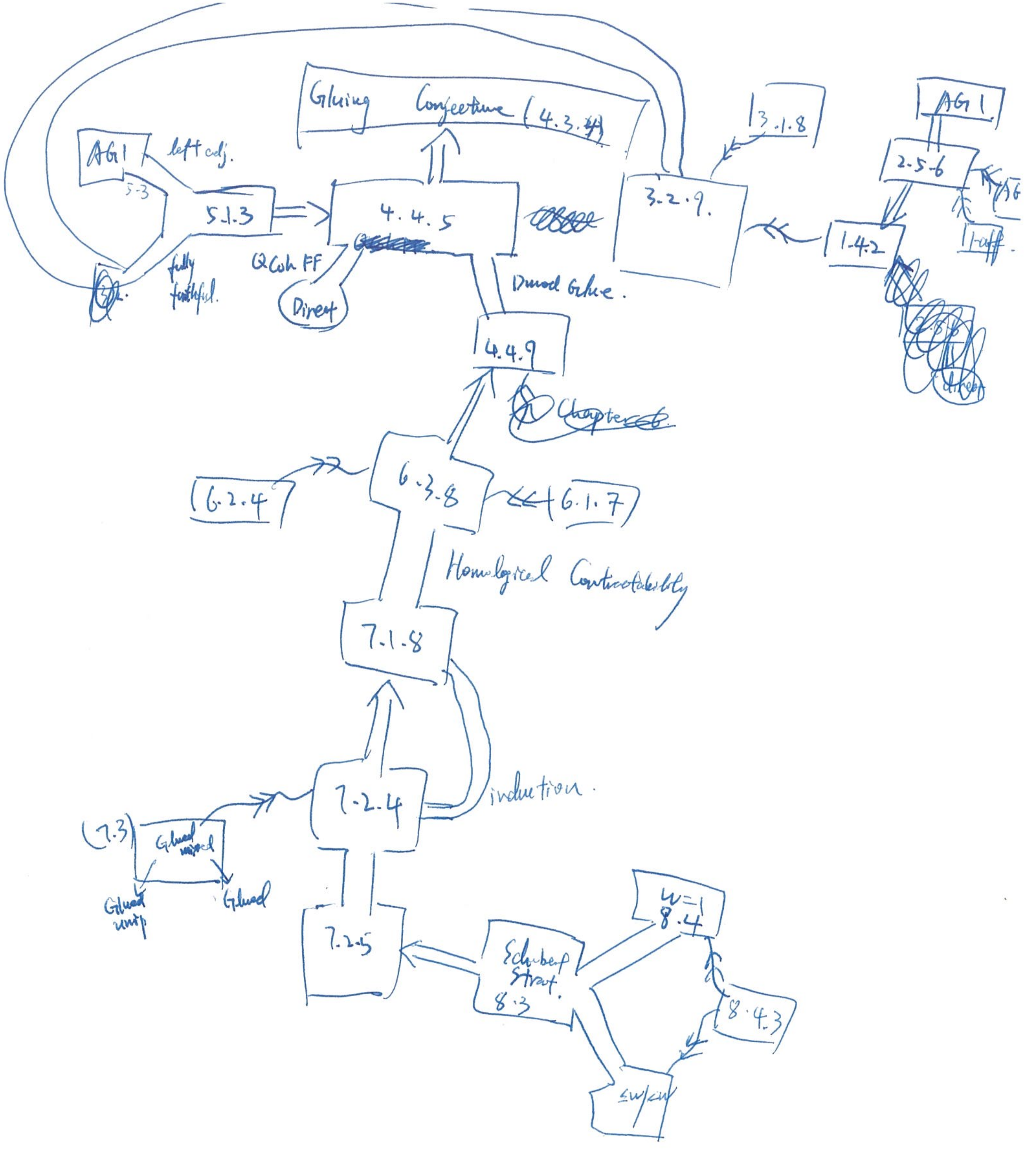

Gluing Conjecture (4.3.4)
AG1
left adj.
5.1.3
4.4.5
QCoh FF
Direct
fully faithful.
Dual Glue.
3.2.9.
3.1.8
AG1
2.5.6
1.4.2
4.4.9
6.2.4
6.3.8
6.1.7
Homological Contractibility
7.1.8
7.2.4
induction.
(7.3
Glued
Glued
7.2.5
Echelon Strat. 8.3
w=1 8.4
8.4.3
sw/cw

The Gluing Conjecture: Overview.

From previous discussions we've learned that $IC_{Nilp}(LocSys_{\check{G}})$ is generated by $QCoh(LocSys_M)$ under Eisenstein, for M ranging over the Levis. In some sense, IC_{Nilp} is therefore a large enough extension.

On the other hand, leaving QCoh does cause a few concerns. Most importantly:

- The spectral localization talks to QCoh not IndCoh.

By which I mean the following. We have the following statement, which holds unconditionally

$\mathbb{LL}^{whit}_{G,G}$:

$$\begin{tikzcd} QCoh(LocSys_{\check{G}}) \arrow[d, "coLoc"] \\ Rep(\check{G})^{unital}_{Ran(X)} \otimes_{\ldots} G] \arrow[d, "\wr\ \text{Casselman-Shalika}"] \\ Whit(G,G). \end{tikzcd}$$

$$\begin{tikzcd} QCoh(LocSys_{\check{G}}) \arrow[r, "\mathbb{LL}^{whit}_{G,G}"] & Whit(G,G) \\ QCoh(Op_{\check{G}})^{glob}_{\lambda I} \arrow[u, "(v_\lambda^I)_*"] \arrow[r, "q\text{-Hitch}_{\lambda I}"] & Dmod(Bun_G) \arrow[u, "coeff_{G,G}"'] \\ QCoh(Op_{\check{G}})^{loc}_{\lambda I} \arrow[u, "(u_\lambda^I)^*"] \arrow[r, "\mathbb{LL}_G^{Op_{\lambda I}}"] & KL(G,crit)_{\lambda I} \arrow[u, "Loc_{G,\lambda I}"'] \end{tikzcd} \quad \boxed{A_1}.$$

This is required to ~~formulate~~ postulate the global GL functor. If we were ambitious, we could try to run the [outline] proof (*) with the following diagram:

$$\begin{tikzcd} Glue^{Fmod}(IC_{Nilp}(LS_{\check{P}})) \arrow[r, "\simeq?", "②"'] & Glue(I(G,P)). \\ IC_{Nilp}(LS_{\check{G}}) \arrow[u, "①"] & D(Bun_G) \arrow[u, "Glue(CT_P^{enh})"'] \end{tikzcd}$$

where ① would potentially be easier! But ② is still conjectural, ~~also~~ so we don't know if this commutes. So we'd be in better shape if we knew that [Done?]

$$IC_{Nilp}(LS_{\check{G}}) \longrightarrow Glue^{Fmod}(QCoh(LS_{\ldots}$$

already.

$$\begin{tikzcd} \cancel{Glue}^{Fmod}(IC_{Nilp}(LS_{\check{P}})) \arrow[r] & \cancel{Glue}(I(G,P)) \\ IC_{Nilp}(LS(\check{G})) \arrow[u] & D(Bun_G) \arrow[u, "\cancel{Glue}(CT_P^{enh})"'] \\ & QCoh(Op(\check{G})^{glob}_{\lambda I}) \arrow[ul, "\Xi \circ v^I_{\lambda *}"] \arrow[u, "q\text{-Hitch}_{\lambda I}"'] \end{tikzcd} \quad \boxed{A_2}$$

$\longrightarrow$

*: recall the [outline] story:

The goal is to prove GL via the following diagram:

$$\begin{tikzcd} \mathrm{Glue}(\text{F-mod}(QC(LS_{\check{P}}))) \arrow[r, "\text{Quasi-Theorem}"] & \mathrm{Glue}(\mathrm{Whit}(G,P)) = \mathrm{Whit}^{ext}_{G,G} \\ IC_{Nilp}(LS_{\check{G}}) \arrow[u, hook, "\boxed{\text{This paper}}"'] & \mathrm{Dmod}(\mathrm{Bun}_G) \arrow[u, "\text{coeff}"'] \end{tikzcd}$$

and by checking the essential images of the generators match.

In other words, we want to embed into QCoh because we want to use Whittaker coeff, which in turn is the case because we want to use local-to-global.

Questions:

1) Can I really not say sth about local-to-global w/ $I(G,P)$?

2) If I were to formulate renormalized GL, what is the corresponding $I(G,P)_{ren}$?

$$\begin{tikzcd} \text{F-mod}(QCoh(LS_{\check{P}})) \arrow[r] & \mathrm{Whit}(G,P) \\ \text{F-mod}(IC_{Nilp}(LS_{\check{P}})) \arrow[u, hook, "\Phi"] \arrow[r, "\sim"] & I(G,P) \arrow[u, "\text{coeff}"'] \end{tikzcd}$$

$\text{F-mod}(IC(LS_{\check{P}}))$?

Let's try to say this more precisely. First let's explain gluing.

Let $(i \mapsto C_i, \alpha \mapsto \Phi_\alpha)$ be a (lax functor) diagram $I \to \mathrm{DGCat}_{cont}$.

Recall by general Lurie, this corresponds to a coCartesian fibration $C_I \to I$, and the limit of the diagram is given by the cat of coCartesian sections $I \to$ (i.e. sections that send 1-morphism to coCartesian morphism.).

If instead we take ***all*** sections, we form lax limit. $\mathrm{laxlim}_{i \in I} C_i$.

Concretely, it consists of $c_i \in C_i$, along with ***morphisms*** $\Phi_\alpha(c_i) \to c_j$.

The universal property is obvious: $D \to \mathrm{laxlim}\, C_i \iff \left\{ \begin{array}{l} (F_i : D \to C_i) \\ (\Phi_\alpha \circ F_i \to F_j) \end{array} \right\}$.

not necessarily iso.

Then we shall write $\mathrm{Glue}(C_i)$ for $\mathrm{laxlim}_{i \in I} C_i$.

Example: $Y_0 \xhookrightarrow[\text{open}]{j} Y \xhookleftarrow[\text{complement}]{i} Y_1$

Y topological.

$C_0 = \mathrm{Shv}(Y_0)$, $C_1 = \mathrm{Shv}(Y_1)$.

$\Phi_{0 \to 1} = i^! \circ j_!$.

Then $\mathrm{Shv}(Y) \to \mathrm{Glue}(C_i)$.

$F \mapsto (j^! F, i^! F, i^! j_! j^! F \to i^! F)$.

and $\mathrm{Glue}(C_i) \to \mathrm{Shv}(Y)$

$(F_0, F_1, i^! j_! F_0 \to F_1) \mapsto \mathrm{Cone}(i_! \ker(i^! j_! F_0 \to F_1) \to j_! F_0)$

are mutual inverses.

IndCoh Let $I^{op} = \mathrm{Par}(G)$. The poset of standard parabolics in G.

First consider $P \mapsto \mathrm{LocSys}_P$, $P \to Q \mapsto \underbrace{\mathrm{LocSys}_P \to \mathrm{LocSys}_Q}_{f_{PQ}}$.

which induces

$P \mapsto \mathrm{IndCoh}\left((\mathrm{LocSys}_P)_{dR} \underset{(\mathrm{LocSys}_G)_{dR}}{\times} \mathrm{LocSys}_G\right)$.

$P \hookrightarrow Q \mapsto (f_{PQ\,dR} \times \mathrm{Id})^!$

Now observe that $\mathrm{Sing}(f_{PQ}) : \mathrm{LocSys}_P \underset{\mathrm{LocSys}_Q}{\times} \mathrm{Sing}(\mathrm{LocSys}_Q) \longrightarrow \mathrm{Sing}(\mathrm{LocSys}_P)$.

sends $(\mathrm{LocSys}_P \underset{LS_G}{\times} \{0\})$ zero section to zero section (duh). Thus $(f_{PQ_{dR}} \times \mathrm{Id})^!$

sends $\mathrm{IndCoh}_{\{0\}}((\mathrm{LocSys}_Q)_{dR} \underset{LS_{G_{dR}}}{\times} LS_G)$ to $\mathrm{IndCoh}_{\{0\}}(LS_{P_{dR}} \underset{LS_{G_{dR}}}{\times} LS_G)$.

Observe that

$$\begin{array}{ccc}
\mathrm{IndCoh}_{\{0\}}(LS_{P_{dR}} \underset{LS_{G_{dR}}}{\times} LS_G) & \longrightarrow & \mathrm{IndCoh}(LS_{P_{dR}} \underset{LS_{G_{dR}}}{\times} LS_G) \\
\downarrow & & \downarrow \text{(pullback along } LS_P \to LS_{P_{dR}} \underset{LS_{G_{dR}}}{\times} LS_G) \\
\mathrm{IndCoh}_{\{0\}}(LS_P) = \mathrm{QCoh}(LS_P) & \longrightarrow & \mathrm{IndCoh}(LS_P)
\end{array}$$

is a pullback diagram, which justifies the notation "$\mathrm{QCoh}(LS_P)_{conn/LS_G}$" appearing in the paper. (So it's "IndCoh on formal neighborhood of image, whose pullback is QCoh".)

<u>Note</u> This category lives in between $\mathrm{QCoh}(LS_{P_{dR}} \underset{LS_{G_{dR}}}{\times} LS_G)$ and $\mathrm{IndCoh}(LS_{P_{dR}} \underset{LS_{G_{dR}}}{\times} LS_G)$

For $Z \to Y$ DG-scheme [Q: does this hold for LS?], we have (to be proven).
quasi-smooth

$$\begin{array}{c}
\mathrm{IndCoh}(Z_{dR} \underset{Y_{dR}}{\times} Y) = \mathrm{QCoh}(Z_{dR} \underset{Y_{dR}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{IndCoh}(Y). \\
\uparrow F \mapsto \Gamma(F) \otimes (i^! \omega_Y) \qquad \cup \\
\mathrm{QCoh}(Z)_{conn/Y}. \\
\cup \\
\mathrm{QCoh}(Z_{dR} \underset{Y_{dR}}{\times} Y) = \mathrm{QCoh}(Z_{dR} \underset{Y_{dR}}{\times} Y) \underset{\mathrm{QCoh}(Y)}{\otimes} \mathrm{QCoh}(Y).
\end{array}$$

and we'll be able to say more about exactly what this middle guy is, once we've introduced the ~~concepts~~ necessary language.

$\longrightarrow$

Also let us connect this w/ the discussion above:

$Z \xrightarrow{p} Z_{dR} \underset{Y_{dR}}{\times} Y$ induces $p^!: IC(Z_{dR} \underset{Y_{dR}}{\times} Y) \longrightarrow IC(Z)$

and its left adj.

$p_*: IC(Z) \longrightarrow IC(Z_{dR} \underset{Y_{dR}}{\times} Y)$.

By general Lurie, we have.

$$IC(Z_{dR} \underset{Y_{dR}}{\times} Y) = (p^! \circ p_*)\text{-mod}(IC(Z)).$$

we shall prove

$$IC_{\mathcal{N}}(Z_{dR} \underset{Y_{dR}}{\times} Y) := (p^!)^{-1}(IC_{\mathcal{N}}(Z)) = (p^! \circ p_*)\text{-mod}(IC_{\mathcal{N}}(Z)).$$

so in particular.

$$IC_{\{0\}}(Z_{dR} \underset{Y_{dR}}{\times} Y) \simeq \underbrace{(p^! \circ p_*)}_{(\text{This is the } F)}\text{-mod}(QCoh(Z)).$$

Back to the main story. We can now form $\mathrm{Glue}\left(IC_{\{0\}}\left(LS_{P_{dR}} \underset{LS_{G_{dR}}}{\times} LS_G\right)\right)$.

and we have a functor

$$\mathrm{IndCoh}_{Nilp}(LS_G) \hookrightarrow \mathrm{IndCoh}(LS_G) \longrightarrow \mathrm{IndCoh}\left(LS_{P_{dR}} \underset{LS_{G_{dR}}}{\times} LS_G\right)$$

$$\Big\downarrow \text{right adjoint to inclusion}$$

$$IC_{\{0\}}\left(LS_{P_{dR}} \underset{LS_{G_{dR}}}{\times} LS_G\right).$$

which combine to a functor $\mathrm{IndCoh}_{Nilp}(LS_G) \longrightarrow \mathrm{Glue}\left(\mathrm{QCoh}(LS_P)_{conn}/LS_G\right)$.

Theorem (4.3.4) This functor is fully faithful.

The rest of this talk is to outline the proof.

The general paradigm is the following ~~xxxxx~~ (Chapter 5).

Recall the lax-limit setup (C_i, Φ_α). Let us be given an orthogonal decomposition of this: $C_i' \subset C_i$ be full subcat, suppose $\Phi_\alpha(C_i') \subset C_j'$, and write $\overset{\circ}{C}_i := (C_i')^{\perp} \subset C_i$ its (right) orthogonal complement.

Suppose further $\Phi_\alpha(\overset{\circ}{C}_i) \subset \overset{\circ}{C}_j$.

Define $C := \mathrm{Glue}(C_i)$, $C' := \mathrm{Glue}(C_i')$, $\overset{\circ}{C} := \mathrm{Glue}(\overset{\circ}{C}_i)$.

Then $\overset{\circ}{C} \subset (C')^{\perp}$. Let $F: D \to C$, corresponding to $F_i: D \to C_i$.

Suppose we're given a compatible decomposition of D:

$$D' \subset D, \quad \overset{\circ}{D} := (D')^{\perp}, \quad \text{s.t. } F_i(D') \subset C_i', \ F_i(\overset{\circ}{D}) \subset \overset{\circ}{C}_i$$

Then they glue to give $F': D' \to C'$. $\overset{\circ}{F}: \overset{\circ}{D} \to \overset{\circ}{C}$.

[equivalent?]

Prop 5.1.3 If 1) F_i all admit left adjoint F_i^L, $F_i^L(\overset{\circ}{C}_i) \subset \overset{\circ}{D}$. (we'll have $F_i^L(C_i') \subset D'$ as well).

2) F' and $\overset{\circ}{F}$ are both fully faithful.

Then F is fully faithful.

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This is general nonsense so let's do it now.

Observe that 1) means F admits a left adjoint F^L s.t. $F^L \circ ins_i \cong F_i^L$.

Namely, (6.2) it is given as follows: ($C = \mathrm{Glue}(C_i, i \in I)$).

Define $\mathrm{String}(I)$ to be the cat w/ obj being $(i_0 \to i_1 \to \cdots \to i_n)$, $i \in I$
and morphisms are those induced from order-preserving $[m] \to [n]$.

Then there's a functor $F^L_{\mathrm{String}} : \mathrm{Glue}(C_i) \longrightarrow \mathrm{Funct}(\mathrm{String}(I), D)$

$$(c_i, \Phi_\alpha(c_i) \longrightarrow c_j) \longmapsto \left(\begin{array}{c} (i_0 \to i_1 \cdots i_n) \\ \downarrow \\ F^L_{i_n}(\Phi_{i_0 \to i_n}(c_{i_0})) \end{array} \right).$$

Compose this w/ colim, we get a map $\mathrm{Glue}(c_i) \longrightarrow D$.

Claim: (6.2.4) This is F^L.

Further (5.1.5) if $F_i^L(\mathring{C}_i) \subset \mathring{D}$, then

$$\begin{tikzcd} C \arrow[d, "F^L"] & \mathring{C} \arrow[l, hook'] \arrow[d, "\mathring{F}^L = \mathrm{Glue}(\mathring{F}_i^L : \mathring{C}_i \to \mathring{D})"] \\ D & \mathring{D} \arrow[l, hook'] \end{tikzcd}$$

commutes.

(One immediately reduce to each C_i, then it's obvious.)

So to show F fully faithful, it suffices to check [How to make precise?].

$$\begin{array}{c} \mathrm{Hom}(\mathring{d}, d') \cong \mathrm{Hom}(\mathring{F}(\mathring{d}), F'(d')). \\ \wr\!\!\mid \ (\text{above}) \\ \mathrm{Hom}(\mathring{F}^L \circ \mathring{F}(\mathring{d}), d') \\ \wr\!\!\mid \ (\mathring{F} \text{ fully faithful}) \\ \mathrm{Hom}(\mathring{d}, d'). \end{array}$$

so the idea is that, if F can be "block decomposed", then to show it's "orthogonal" on the entire space it suffices to check on each block.

The way we're going to instantiate this is by setting.

$$D = \mathrm{IndCoh}_{Nilp}(\mathrm{LocSys}_G). \quad D' = \mathrm{QCoh}(LS_G).$$

$$\mathring{D} = \mathring{IC}(LS_G) \cap IC_{Nilp}(LS_G), \text{ where } \mathring{IC}(LS_G) = IC(LS_G)/\mathrm{QCoh}(LS_G)$$

$$\triangleq \mathring{IC}_{Nilp}(LS_G).$$

(eqv. the right orthogonal $\boxed{?}$ complement of $\mathrm{QCoh}(LS_G)$).

$$C_i = \mathrm{IndCoh}_{\{0\}}(LS_{P_{dR}} \underset{LS_{G_{dR}}}{\times} LS_G).$$

$$C_i' = \mathrm{QCoh}(LS_{P_{dR}} \underset{LS_{G_{dR}}}{\times} LS_G).$$

$$\mathring{C}_i = \Big(\mathrm{QCoh}(LS_{P_{dR}} \underset{LS_{G_{dR}}}{\times} LS_G) \underset{Q(LS_G)}{\otimes} \mathring{IC}(LS_G)\Big) \cap \mathrm{IC}$$

$$\triangleq \mathring{IC}_{\{0\}}(LS_{P_{dR}} \underset{LS_{G_{dR}}}{\times} LS_G)$$

and $F_i : \mathrm{IndCoh}_{Nilp}(LS_G) \hookrightarrow \mathrm{IndCoh}(LS_G) \xrightarrow{p^!} \mathrm{IndCoh}(LS_{P_{dR}} \underset{LS_{G_{dR}}}{\times} LS_G) \xrightarrow{(\mathrm{all} \to \{0\})} C_i.$

The rest of Chapter 5 of the paper is then devoted to checking that these assumptions do hold. The difficulty lies in checking the fully faithfulness of the IndCoh/QCoh part, i.e. that of the functor $\mathring{D} \to \mathring{C}$.

This relies on the theory developed in Chapter 1~3, namely, we'll learn that:

[INSERT: Proj Supp (Back)]

(1.4.2) For Z quasi-smooth scheme, There exists an canonical object $\mathring{IC}(Z)^\sim \in \mathrm{ShvCat}((\mathbb{P}\mathrm{Sing}\, Z)_{dR})$.

such that for any $N \subset \mathbb{P}\mathrm{Sing}(Z)$, we have

$$\Gamma(N_{dR}, \mathring{IC}(Z)^\sim) = \mathring{IC}_N(Z).$$

(further, this is compatible w/ the $\mathrm{QCoh}(Z_{dR})$ action.)

Proj supp:

Recall that $QCoh(Z) = IndCoh_{\{0\}}(Z)$. Thus, for each $\mathcal{F} \in IC(Z)/QCoh(Z)$, it makes sense to assign to it its projective singular support $\operatorname{supp} \mathcal{F} \subset \mathbb{P}Sing(Z) := (Sing(Z) - \{0\})/\mathbb{G}_m$.

and we can define for any conical $\mathcal{N} \subset \mathbb{P}Sing(Z)$:

$$\mathring{IC}_{\mathcal{N}}(Z) = \{\mathcal{F} \in \mathring{IC}(Z) \mid \mathbb{P}SingSupp(\mathcal{F}) \subset \mathcal{N}\}.$$

$$= \mathring{IC}(Z) \cap IC_{\mathcal{N}}(Z).$$

What this buys us is the ability, modulo some $\widehat{\text{easy}}$ 1-affine observation, to write

$\mathrm{Sing}(f): Z \underset{Y}{\times} \mathrm{Sing}(Y) \to \mathrm{Sing}(Z)$

$$(3.2.9) \qquad \mathrm{IndCoh}^{\circ}_{N}(Z_{dR} \underset{Y_{dR}}{\times} Y) \simeq \Gamma\left(\mathbb{P}(\mathrm{Sing}\, f)^{-1}(N))_{dR}, \overset{\circ}{\mathrm{IC}}(Y)^{\sim}\right)$$

$$\simeq \mathrm{Dmod}(\mathbb{P}((\mathrm{Sing} f)^{-1}(N)) \underset{\mathrm{Dmod}(\mathbb{P}\mathrm{Sing}\, Y)}{\otimes} \overset{\circ}{\mathrm{IC}}(Y).$$

Two extrema:

$$\overset{\circ}{\mathrm{IC}}(Z_{dR} \underset{Y_{dR}}{\times} Y) = \mathrm{Dmod}(Z \underset{Y}{\times} \mathbb{P}\mathrm{Sing}\, Y) \underset{D(\mathbb{P}\mathrm{Sing}\, Y)}{\otimes} \overset{\circ}{\mathrm{IC}}(Y).$$

$$\overset{\circ}{\mathrm{IC}}_{\{0\}}(Z_{dR} \underset{Y_{dR}}{\times} Y) = \mathrm{Dmod}(\mathbb{P}(\mathrm{Sing} f)^{-1} 0)) \underset{D(\mathbb{P}\mathrm{Sing}\, Y)}{\otimes} \overset{\circ}{\mathrm{IC}}(Y).$$

In other words, the "purely singular" part of our problem localizes "topologically" (i.e. it is local over the de Rham spaces.) By general nonsense, the fully faithfulness of $\overset{\circ}{D} \to \overset{\circ}{C}$ reduces to that of

$$F: \mathrm{Dmod}(\mathbb{P}\mathrm{Nilp}) \longrightarrow \mathrm{Glue}(\mathrm{Dmod}(\mathbb{P}(\mathrm{Sing} f_i)^{-1}(\{0\})))$$

which is now a topological problem.

The remaining Chapters (6 ~ 9) are devoted towards this statement.

~~Recall the left adjoint ... pullback along~~ $\cancel{\mathbb{P}(\mathrm{Sing} f_i^{-1}(\{0\})) \to \mathbb{P}(LS_P \times_{LS_G} \mathrm{Sing}(LS_G))}$. Observe that each F_i is given by

$$D(\mathbb{P}(\mathrm{Nilp})) \xrightarrow{j_*} D(\mathbb{P}\,\mathrm{Sing}\, LS_G) \xrightarrow{p^!} D(LS_P \underset{LS_G}{\times} \mathbb{P}\,\mathrm{Sing}\, LS_G)$$

$$\xrightarrow{q^!} D(\mathbb{P}\,\mathrm{Sing} f^{-1}\{0\})$$

Let F^L denote its left adjoint. By the "String" description above, we can observe.

$$F^L \circ F \simeq (f_{Glued})_{dR,!} \circ (f_{Glued})^!_{dR}$$

where $f_{Glued} : \mathcal{M}_{Glued} \to \mathbb{P}\,Nilp_{glob}$, $\mathcal{M}_{Glued}$ is the colimit over $Strings(Par_G)$ of $(P_0 \subset P_1 \subset \cdots P_n) \longmapsto \left(LoS_{P_0} \underset{LS_G}{\times} \mathbb{P}Sing\,LS_G\right) \times \mathbb{P}Sing f^{-1}\{0\}$.

$$\left(LS_{P_n} \underset{LS_G}{\times} \mathbb{P}Sing\,LS_G\right)$$

where $(f_{Glued})_{dR,!}$ is the left adjoint of $(f_{Glued})^!_{dR}$, defined because the RHS in the expression above is proper over $\mathbb{P}\,Nilp_{glob}$.

So we reduce to $(f_{Glued})^!_{dR}$ being fully faithful. (i.e. f_{Glued} is homologically contract
By general theory, this is equivalent to fiber being homologically trivial. The fiber is the same colimit as above, where we take fiber on the RHS.

Let us further describe this fiber. Fix (σ, A) a k-point of $Nilp_{glob}$.
i.e. σ is a G-local system on X, A is a nilpotent section of $\mathfrak{g}_\sigma^* \simeq \mathfrak{g}_\sigma$.
(the coad-adjoint bundle).

We write $Spr^{\sigma,A}_{Glued,unip} = \underset{Strings(Par_G)}{colim}\; Spr^\sigma_{P_0} \underset{Spr^\sigma_{P_n}}{\times} Spr^{\sigma,A}_{P_n,unip}$. $(P_0 \subset P_n)$
where RHS classifies reduction of σ to P_0, such that
A is a section of $Unipotent(P_n)_\sigma \subset \mathfrak{g}_\sigma$.

Claim: This is the fiber at σ.

Then we reduce to: (writing $C_*(Z) := f_{dR,!}(\omega_Z)$) $f : Z \to pt$

Thm 7.1.8 $\quad C_*(Spr^{\sigma,A}_{Glued,unip}) \simeq k$

In Chapter 7, by parabolic induction, this is further reduced to

$$(7\text{-}2\text{-}5) \qquad C_*\left(Spr^{\sigma,A}_{\text{Glued}}\right) \simeq k. \quad \text{where } Spr^{\sigma,A}_{\text{Glued}} = \underset{\substack{\text{Proper}\\ \text{Parabolic}}}{\text{colim}}\ Spr^{\sigma,A}_P.$$

where $Spr^{\sigma,A}_P$ is a reduction of σ to P s.t. A is a section of $P_\sigma \subset g_\sigma$.

In Chapter 8 this is proven. The way this is done is as follows:

Recall that A, being a nilpotent section, fixes a reduction to a standard parabolic. (Namely, at any point w/ value e, complete it to a sl_2-triple (e,f,h) and take p to be the positive eigenspaces of h). By considering the difference between this and the given reduction specified by a pt on $Spr^{\sigma,A}_P$, we can give a filtration by W', where $W' = \{w : w^{-1}(\text{the roots corresponding to } P_0) \subset \text{positive}\}$. (note it contains maximum element w_0', throwing everything else to the negative)

This filtration also holds for the glued version. Then we show that the "associated graded" is trivial.