Goal: to access singular support more easily (concretely).

Key observations:

1) For C ~~nice~~ sym. monoidal, $c \in C$ compact generator, we have $C \simeq \mathrm{End}(c)\text{-mod}$.

$$M \longmapsto \mathrm{Hom}(c, M) \qquad \text{(Functorial)}.$$

$$c \underset{\mathrm{End}(C)}{\otimes} N \longleftarrow N$$

In particular, if $\mathbb{1}$ is compact generator, then $C \simeq HC(C)\text{-mod}$.

(namely, $C = \mathrm{IndCoh}(X)$) for $X = pt \underset{Y}{\times} pt$.

2) Support of modules $\leftrightsquigarrow$ Sing supp of sheaf.

3) In nice case, $HC(C)$ is particularly easy.

Applications:

1) ~~Compact~~ ~~Methods~~ ~~generators~~ Generators of IndCoh_Y.

2) $\mathrm{QCoh} = \mathrm{IndCoh}_0$.

We observed $\subset$, so need converse.

$HC_0\text{-mod} \xrightarrow{\sim} \mathrm{IndCoh}_0 \hookrightarrow \mathrm{IndCoh}$

~~①~~ Reinterpret the composite ~~functor~~

then show it's ~~compact~~ generators lie in image of $\simeq$

3) Geometric Satake. (no time).

General Background.

A. By groupoid we mean simplicial objects $\cdots R \underset{X}{\times} R \substack{\rightarrow\\\rightarrow\\\rightarrow} R \rightrightarrows X$
$\mathrm{Grpoid}(X)$

and write
$$\begin{array}{ccc} & R & \\ \swarrow & & \searrow \\ X & & X \end{array}$$
for simplicity.

initial obj: X (diagonal grpd)

final obj: $X \times X$ (free grpd)

For this talk!
↓
(really, R, X both affine schemes).

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ X & X \arrow[u, "\text{unit}"] & X \end{tikzcd}$$

In this talk further: ~~add~~ $R \xrightarrow{\text{target}} X$ and $R \underset{X}{\times} R \xrightarrow{m} R$ are schematic

Then 1) $\mathrm{IndCoh}(R)$ admits monoidal structure.

$$\begin{tikzcd} R \underset{X}{\times} R \arrow[r] \arrow[d] & R \times R \\ R & \end{tikzcd}$$

Example: free groupoid $X G S$
$\rightsquigarrow \mathrm{IndCoh}(X \times X)$
$\simeq \mathrm{Funct}_{\mathrm{cont}}(\mathrm{IndCoh}(X), \mathrm{IndCoh}(X))$.

2) $\mathrm{IndCoh}(R) \curvearrowright \mathrm{IndCoh}(X)$

$$\begin{tikzcd} R \arrow[r, "s \times id"] \arrow[d, "t"'] & X \times R \\ X & \end{tikzcd}$$

Thus $HC(\mathrm{IndCoh}(R))$ is an E_2-algebra.
$= \mathrm{End}(\mathrm{unit}_*^{IC}(\omega_X))$.

(Remark: use QCoh gives the same E_2-alg)

B. Recall the general phenomenon explained in WEU seminar:

$\mathrm{LMod}_\bullet(C) : E_n\text{-alg}(C) \longrightarrow E_{n-1}\text{-alg}(C\text{-ModCat})$
is fully faithful.

More precise ver: $\exists$ fully faithful sym monoidal

$E_1\text{-alg} \longrightarrow \text{pointed } \mathrm{DGCat}_{\mathrm{cont}}$.
$A \longmapsto (A, A^{op}\text{-mod})$.

essential image:
(c, C)
C compactly generated by c

Right adj:

$(c, C) \longmapsto \mathrm{End}_C(c, c)$.

right-lax $\rightsquigarrow$ upgrade to map of algebras.

$E_1\text{-alg}(\mathrm{DGCat}) \longrightarrow E_2\text{-alg}$

Counit: $\mathcal{O} \rightsquigarrow (\mathbb{1}_{\mathcal{O}}, \mathcal{O}) \rightsquigarrow \mathrm{End}_{\mathcal{O}}(\mathbb{1}_{\mathcal{O}}) \in E_2\text{-alg}$

$$\begin{tikzcd} \underbrace{\mathrm{End}_{\mathcal{O}}(\mathbb{1}_{\mathcal{O}})^{\text{int-op}}\text{-mod}}_{} \arrow[d] \\ \mathcal{O} \end{tikzcd}$$

General nonsense $\Rightarrow$

~~Claim~~ iso iff $\mathbb{1}_{\mathcal{O}}$ compact generator.
(fully faithful $\Rightarrow$ idempotent)

This is something: a nice sym. monoidal cat can be recovered from its center!

What's a good example? $\mathcal{V}[-1]$ for $\mathcal{V}$ affine smooth scheme.

Any quasi-smooth $X^{cl} = pt$ looks like this!

i.e.

$$\begin{tikzcd} & \mathcal{V}[-1] \arrow[dl] \arrow[dr] & \\ pt & & pt \end{tikzcd}$$

this is a group scheme $\boxed{\text{Add} : \mathcal{V}[-1]}$.

See page 13

Let $\Delta_{pt} : pt \to pt \times_{\mathcal{V}} pt$ be the diagonal. $= \mathcal{V}[-1]$. (This is closed embedding.)

($\mathcal{V} = V$: Spec Sym $V^*[1]$).

The $\mathbb{1}$ here is $(\Delta_{pt})_*^{IC}(k)$. Claim:

1) This generates $IC(\mathcal{V}[-1])$ (because underlying classical reduced is a pt — 4.1.8).
2) This is compact. (need: IndCoh$(S^0) = 0$?)

Then story above yields:

$$\text{IndCoh}(\mathcal{V}[-1]) \simeq \underline{HC(pt/\mathcal{V})}\text{-mod}.$$
$$\triangleq \text{End}(\Delta_{pt*}\Delta^! k)$$

So we're back to our sing. supp. setup. To proceed, we need to compute $H^\bullet HC(pt/\mathcal{V})$. This is given by:

$$HC(pt/\mathcal{V}) \simeq \text{Sym}(V[-2]), \quad V = T_{pt}\mathcal{V}.$$

That is to say, $\text{Spec}(HH) = V^*$. $\boxed{\text{as } E_1\text{-algebras!}}$

I need to explain why this is the case.
The general statement is that:

G grp. if $\begin{matrix} G \\ Z \end{matrix}$ affine, $\begin{tikzcd} & G \arrow[dl] \arrow[d] \arrow[dr] & \\ Z & Z \arrow[u] & Z \end{tikzcd}$, $T^*(G/Z)$ perfect.

Then $HC(\text{IndCoh}(G)) \simeq \Gamma(Z, \text{Sym}_{\mathcal{O}_Z}(L_G))$ $\boxed{\text{as } E_1!}$

$$L_G \triangleq \text{unit}^*(T(G/Z))[-1].$$
$\uparrow$ QCoh(Z).

(Instantiation: see back)

$$T_{X/Y} \longrightarrow T_{X/Z} \longrightarrow f^* T_{Y/Z}.$$

$T_G = V[-1]$.

Namely, $\mathrm{unit}^* \big(T(G/pt)[-1] \big)$ ☆

$= T(pt/G)$

$= [k \to 0 \to V]$.

☆: $\mathrm{unit}^* \big(T(G/Z) \big)[-1] = T(Z/G)$. $(X = Z,\ Y = G)$.

This follows from two observations:

1) $L_G \in QCoh(Z)$ has a Lie alg structure.

1a) This structure is trivial when G is grp.

2) $\exists$ $T(Z, U_{O_Z}(L_G)) \longrightarrow HC(IndCoh(G))$

which is iso when Z is eventually coconn.

use this

(G.2.2)

General setup:

$i: Z \longrightarrow W$ proper between affine DG scheme. $W = G$.

$s: W \to Z$ s.t. $s \circ i = id_Z$. $i: Z \to G$

$T^*(Z/W)$ perfect. $s = p_1$.

$\rightsquigarrow T(Z/W) \in QCoh(Z)^{perf}$.

then: 1) $T(Z/W)$ has Lie structure.

2) $\underbrace{T}_{\boxed{\text{as } E_1}}(Z, U_{O_Z}(T(Z/W)) \longrightarrow HC(IndCoh(W))$.

eventually connective.

This is because

$L_G \simeq \Omega(unit^*(T(G/Z)))$.

and loops of Lie are trivial.

$Z \times_G Z \simeq \Omega(G)$. as grp $/Z$.

G.2.2's proof:

(Start w/ ✡)

~~$i: Z \to W$, $s: W \to Z$.~~

$Z \times_W Z$ $\quad$ $i: Z \underset{s}{\rightleftarrows} W$

~~1) $i^! \circ i_*^{IC} \simeq p_{2*}^{IC} \circ p_1^!$~~

~~Base change.~~

~~2) Thus $End(i_*^{IC}(F)) \simeq Hom(F, p_{2*} p_1^!(F))$.~~

✡ : retraction $\Rightarrow$ $Z \times_W Z$ is group over Z.

In general: $unit^*(T^*(G/Z)) \in coLie(QCoh(Z))$.

AFT: eventually connective, w/ coherent cohomology.

$\in QCoh(Z)^-$

grp scheme in $QCoh^{perf}$

$G = Z \times_W Z.$

$u^* \text{ of } T^{IC}(G/Z) := D^{Serre}(T^*(G/Z)) \in Lie(IC(Z)).$

(In fact in Perf so) $u^* \text{ of } T(G/Z) := D^{naive}(T^*(G/Z)) \in Lie(QCoh(Z)).$ $\otimes \omega_Z$. (Υ_Z)

1a) $u^* T(Z \times_W Z / Z) = T(Z/W).$

follows (usual tangent computation).

2) Base change: $i^! \circ i_*^{IC} = p_{2*}^{IC} \circ p_1^!$

$$\begin{tikzcd} Z \times_W Z \arrow[r] \arrow[d] & Z \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

Thus

$$\mathrm{Hom}_{IC(Z)}(\omega_Z, p_{2*} \underbrace{p_1^! \omega_Z}_{\omega_G}) \overset{\text{as } G_1\text{-alg}}{=} \mathrm{End}(i_* \omega_Z).$$

$p: G \to Z : u$ is iso on underlying scheme

$$\wr|$$

$$\mathrm{Hom}_{IC(Z)}(\omega_Z, p_{2*} \omega_G^\wedge)$$

$\omega_Z^\wedge \in IndCoh(G)_Z$ (set-th. supp)

$IC(G)_Z \underset{\wedge}{\rightleftarrows} IC(G).$

$$\wr|$$

general: $\boxed{Distr(\exp(L)) \simeq U(L).}$

$$\mathrm{Hom}_{IC(Z)}(\omega_Z, U(T^{IC}(G/Z)))$$

Υ_Z fully faithful when Z eventually coconnective.

$$\Gamma(Z, U(unit^*(T(G/Z))[-1]))$$

$$\simeq \mathrm{Hom}_{QC(Z)}(\mathcal{O}_Z, U_{\mathcal{O}_Z}(T(G/Z)))$$

$$\xrightarrow{\Upsilon_Z} \mathrm{Hom}_{IC(Z)}(\omega_Z, U(T^{IC}(G/Z)))$$

as ~~[illegible]~~ ass. alg.

Back to main story. We know how $\cancel{HH}\ HC(\mathcal{Y}[-1]) = \mathrm{Sym}^{op}(\cancel{V}\, V[-2])$

write KD : $IC(\mathcal{Y}[-1]) \longrightarrow HC(pt/\mathcal{Y})$-mod

$\mathcal{M} \longmapsto \mathrm{Hom}(\Delta_{pt*}k, \mathcal{M})$

Now we can define ~~[illegible]~~ support. (i.e. via $HC(pt/\mathcal{Y}) \curvearrowright IC(\mathcal{Y}[-1])$)

Claim: support defined in this way is that of the support of the $\mathrm{Sym}(V[-2])$-mod $H^\bullet KD(-)$ gr.

(Detail on pg. 13)

Pf: (3.4.4) (compactly generated case can be checked on obj.)

Remark: 1) This is also the SS of the sheaf.

2) $IC(\mathcal{Y}[-1]) \simeq \mathrm{Sym}\, V[-2]$-mod right now is not monoidal.. need parallelization (because $HC \simeq \mathrm{Sym}$ is not E_2 yet) (comy up

~~Back page~~

Now there's many functorial properties we can prove ... or skip. (5.2)

General Picture: Z q-smooth.

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d] & U \arrow[d] \\ pt \arrow[r] & V \end{tikzcd}$$

U, V smooth affine.

$$\begin{tikzcd} & Z \times_U Z \arrow[dl] \arrow[dr] \arrow[d] & \\ Z \arrow[d] & pt \times_V pt \arrow[dl] \arrow[dr] & Z \arrow[d] \\ pt & & pt \end{tikzcd}$$

upgrades to

$$\begin{tikzcd} & Z \times_U Z \arrow[dl] \arrow[dr] \arrow[d, "\wr"] & \\ Z \arrow[dd] & (pt \times_V pt) \times Z \arrow[dl, "p_2"'] \arrow[d, "p_1"] \arrow[r, "act"] & Z \arrow[d] \\ & pt \times_V pt \arrow[dl] \arrow[dr] & Z \arrow[d] \\ Z \arrow[d] & & pt \\ pt & & \end{tikzcd}$$

Observe next that we have maps

$$\begin{array}{ccc} Z \times_U Z & & Z \\ \downarrow & & \downarrow \\ (pt \times_V pt) \times U & & IC(pt \times_V pt) \otimes QCoh(U) \\ & & \downarrow \wr \\ & & IC(pt \times_V pt) \otimes IC(U) \\ & & \downarrow {\scriptstyle id \otimes \iota^{*!}} \\ & & IC(pt \times_V pt) \otimes IC(Z) \\ & & \downarrow \wr \\ & & IC(Z \times_U Z). \end{array}$$

hence E_2-map of the center

$$HC(pt/_V) \otimes T(U, O_U) \longrightarrow HC(Z \times_U Z) \to HC(Z).$$

which will provide us ~~the~~ another notion of support of $F \in IC(Z)$ in $\bigoplus H^{2n}(HC(pt/_V) \otimes T(U, O_U)) = Sym(V) \otimes T(U, O_U)$.

~~(The choice of using U is arbitrary we could say it as:~~

$$\cancel{act : (pt \times_V pt) \times Z \to Z.}$$

or rather its Spec, namely $V^* \times U$.

$$\cancel{F \in IC(Z) \rightsquigarrow act^!(F) \in IC(pt \times_V pt) \otimes IC(Z).}$$

~~space~~

~~then~~ $\cancel{act^! F}$ ~~has~~

Main Claim:

This support is $SingSupp(F) \subset Sing(Z) \hookrightarrow V^* \times Z \hookrightarrow V^* \times U$.

Reminder: $Sing = {}^{cl}(Spec_Z(Sym_{O_Z}(T(Z)[1])))$
$= Spec_{cl_Z}(Sym_{O_{cl_Z}}(H^1(T(Z)[1])))$

and $Sing(Z) \hookrightarrow V^* \times Z$ from $T(Z) \longrightarrow \iota^* T(U) \longrightarrow V \otimes O_Z$.
which in turn comes from pulling sect of $T(U \to V)$ along $Z \to U$.

Namely, one needs to check that

$$\begin{tikzcd} \mathrm{Sym}(V) \otimes \Gamma(U, \mathcal{O}_U) \arrow[r] \arrow[dr] & \Gamma(\mathrm{Sing}\, Z, \mathcal{O}) \arrow[d] \\ & HH^{even}(Z) \end{tikzcd}$$

$\Gamma(U, \mathcal{O}_U)$, $\circlearrowleft$

RHS induced by $\Gamma(Z, \mathcal{O}_{cl\,Z}) \to HH^0(Z)$

and $\Gamma(Z, H^1(T(Z))) \to HH^2(Z)$.

$$\nearrow$$

$\Gamma(V \otimes \mathcal{O}_{cl\,Z})$ coming from the maps above.

$$\nearrow$$

$\Gamma(V \otimes \mathcal{O}_U)$

Now recall if A is E_2-alg, $A \circlearrowright$ DGCat C, and

$Y \subset \mathrm{Spec}(\bigoplus H^{2n} A)$ is such that complement is qc, then

$\left(C_Y \simeq A\text{-mod}_Y \otimes_{A\text{-mod}} C\right)$.

Noetherian condition to make sure $T_Y \to A$ has continuous right adjoint

So in particular $\otimes$.

$$IC_Y(Z) \simeq IC(Z) \otimes_{HC(pt/g)\text{-mod} \otimes QC(U)} \left(HC\text{-mod} \otimes QC(U)\right)_Y.$$

Parallelization

~~Let V~~ Recall $V[-1] = \mathrm{Spec}\, \mathrm{Sym}\, V^*[1]$ noncanonically.

Now fix one such iso (a parallelization).

Claim: this upgrades $HC(V[-1]) \simeq \mathrm{Sym}\, V[-2]$ as commutative algebras!

Corollary: $HC\text{-mod} \simeq \mathrm{Sym}\, V[-2]\text{-mod}$ as symmetric monoidal cat.

(see: shifting game) page 12

We proceed. ~~Note QCoh(V/Gm) ... Sym V[-2]-mod~~

The instantiation is $\mathrm{Sym}(V[-2])^{shift} = \bigoplus \mathrm{Sym}\, V$.

Then $S_A = V^*/\mathbb{G}_m$.

$$SG1 \Rightarrow IC(V[-1])_Y = IC(V[-1]) \otimes_{Q(S_A)} Q(S_A)_Y/\mathbb{G}_m.$$

Similarly, for the Z-relative case:

instantiation is $\mathrm{Sym}(V[-2]) \otimes \Gamma(U, \mathcal{O}_U)$.

shifted: $\mathrm{Sym}(V) \otimes \Gamma(U, \mathcal{O}_U)$.

$S_A = V^*/\mathbb{G}_m \times U$.

$$SG \models IC_Y(Z) = IC(Z) \otimes_{QCoh(S_A)} QCoh(S_A)_{Y/\mathbb{G}_m} \quad \text{for } Y \subset \mathrm{Sing}(\ldots)$$

Application

1) Generators of $IC_Y(Z)$.

There's a proper action map

$act : \mathcal{V}[-1] \times Z \longrightarrow Z$ mentioned before.

$$F := act_*^{IC} : IC(\mathcal{V}[-1] \times Z) \rightleftarrows IC : act^! =: G.$$
$$= IC(\mathcal{V}[-1]) \otimes IC(Z)$$

G admits a retract (by pullback along unit) thus conservative thus $\mathrm{im}(F)$ generates.

Another description of F and G:

$$IC(\mathcal{V}[-1]) \otimes IC(Z) \simeq HC(\mathcal{V}[-1])^{op}\text{-mod} \otimes IC(Z)$$
$$\simeq HC(\mathcal{V}[-1])^{op}\text{-mod}(IC(Z)).$$

One hand: $F \in IC(Z)$ is a $HC(Z)^{op}$-mod

$HC(\mathcal{V}[-1]) \to HC(Z/\mathcal{U}) \to HC(Z)$ so

get $HC(\mathcal{V}[-1])^{op}$-mod structure. $\boxed{G}$

Other hand: $\mathcal{V}[-1] \curvearrowright Z$

thus $HC(\mathcal{V}[-1])\text{-mod} \otimes IC(Z) \xrightarrow{act} IC(Z)$

$\boxed{F}$.

Now we interact w/ SS. We specify what it means to have $Y \subset V^* \times \mathcal{U}$ support for $F \in IC(\mathcal{Y}[-1]) \otimes IC(Z)$.
by giving a map of E_2-alg's

$$HC(\mathcal{Y}[-1]) \otimes \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}}) \longrightarrow HC(IC(\mathcal{Y}[-1] \times Z))$$

This comes from

$$\begin{array}{c} \underset{\text{cat}}{IC(\mathcal{Y}[-1])} \otimes \underset{\text{cat}}{QCoh(\mathcal{U})} = \underbrace{(HC \otimes \Gamma)}_{\text{as } E_2\text{-alg}}\text{-mod} \\ \downarrow \\ IC(\mathcal{Y}[-1]) \otimes IC(Z). \end{array}$$

F is compatible w/ $IC(\mathcal{Y}[-1]) \otimes Q(\mathcal{U})$-action on everything
G is lax-compatible but actually compatible.
(key: the action cat is rigid.)

Then it follows formally that

$$F : (IC(\mathcal{Y}[-1]) \otimes IC(Z))_Y \rightleftarrows IC_{Y \cap \mathrm{Sing}\, Z}(Z)$$

compatible with the $\Xi_Z^{Y,\mathrm{all}}$ functors.

$Y \subset \mathrm{Sing}(Z)$ then

Now $\forall\, F \in IC_Y(Z) \iff \Xi_Z^{Y,\mathrm{all}}(F) \xrightarrow{\sim} F$

$\iff G(\text{—}) \xrightarrow{\sim} G(F)$ conservativity

$\iff G(F) \in IC(\mathcal{Y}[-1] \times Z)_Y$.

and F (restricted) is left adj to G conservative, so

$F(IC(\mathcal{Y}[-1] \times Z)_Y)$ generates $IC_{Y \cap \mathrm{Sing}(Z)}(Z)$.

In particular $IC_Y(Z)$ ~~∅~~ since $Y \subset \mathrm{Sing}(Z)$.

Application in real life: Z aff, q-smooth.
then $\Xi_Z : Q(Z) \to IC(Z)$ image is $IC_{\{0\}}(Z)$.
Local, so assume

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & U \arrow[d] \\ pt \arrow[r] & V \end{tikzcd}$$

Suffices to check $Fon(IC(\gamma[-1] \times Z)_{\{0\}\times U})$ has image contained in $\mathrm{im}(\Xi)$.
Further: $p_\gamma : \gamma[-1] \to pt$. claim: $(p_\gamma \times id)^! : IC(Z) \to IC(\gamma[-1] \times Z)$ lands in $(---)_{\{0\}\times U}$ and generates.
This is because (generally)

$$IC(\gamma[-1] \times Z)_{0\times U} = IC(\gamma[-1])_0 \times IC(Z).$$

so suffices to check

$$p_\gamma^! : IC(pt) \to IC(\gamma[-1]) \text{ lands in } IC_0 \text{ and}$$

generates. This can be done by hand or follows formally.

So checking image of $IC(Z) \xrightarrow{(p_\gamma\times id)^!} \xrightarrow{F} IC(Z)$.
Claim: this functor is $\iota^! \circ \iota_*$, $\iota : Z \to U$

$$\begin{tikzcd} Z \times_U Z \arrow[r] \arrow[d] & Z \arrow[d] \\ Z \arrow[r] & U \end{tikzcd}$$

base change. So check image of $\iota^!$.

well, U smooth, so $Q(U) \ni \omega_U$ generates $IC(U)$, so check $\underbrace{\iota^!(\omega_U)}_{=\omega_Z} \in \mathrm{im}(\Xi)$.

$\Downarrow$

Gorenstein.

(Content on back side)

shifting game.

Let $C \circlearrowleft \mathbb{G}_m$. Inv: $\mathbb{G}_m\text{-mod} \underset{\sim}{\overset{\text{(inverses)}}{\rightleftarrows}} \mathrm{Rep}(\mathbb{G}_m)\text{-mod}$: ind.

Observe $\mathrm{Rep}(\mathbb{G}_m)$ admits an auto:

$$M \mapsto M^{shift}, \quad M^{shift}_{(i,k)} = M_{i+2k,k} \quad (\mathrm{Coh}, \mathbb{G}_m).$$

induces $\mathrm{Rep}(\mathbb{G}_m)\text{-mod} \xrightarrow{\sim} \mathrm{Rep}(\mathbb{G}_m)\text{-mod}$

$$C \mapsto C^{shift}. \quad \text{(commutes w/ forgetful)}$$

- Example: A: $\mathbb{Z}$-graded. $\left((A\text{-mod})^{\mathbb{G}_m}\right)^{shift} \cong (A^{shift}\text{-mod})^{\mathbb{G}_m}$.

$$(A\text{-mod})^{\mathbb{G}_m} \xrightarrow{\text{as plain DG}} M^{shift}, \quad M \mapsto$$

Warning: for $\mathbb{G}_m$-mod, this shifting is not obl. to DGCat: $(A\text{-mod})^{shift} = A^{shift}\text{-mod}$

Suppose $A \in E_2\text{-alg}(\mathrm{Rep}(\mathbb{G}_m))$ and A^{shift} is classical. Then it is E_∞ thus so is A.

(shift is sym. monoidal)

By above, we have

$$(A\text{-mod})^{\mathbb{G}_m} = (A^{shift}\text{-mod})^{\mathbb{G}_m} = \mathrm{QCoh}(\overbrace{\mathrm{Spec}(A^{shift}/\mathbb{G}_m)}^{S_A})$$

Next, observe A^{shift} classical $\Rightarrow A^{shift} \simeq \bigoplus H^{2n}(A)$ as classical.

Then $Y \subset \mathrm{Spec}(\bigoplus H^{2n} A)$ conical

$$\Rightarrow Y/\mathbb{G}_m \hookrightarrow S_A \text{ makes sense.}$$

SG1 Prop.

$$A\text{-mod}_Y = A\text{-mod} \underset{QC(S_A)}{\otimes} QC(S_A)_{Y/\mathbb{G}_m}.$$

where $A\text{-mod} = A\text{-mod} \underset{(A\text{-mod})^{\mathbb{G}_m}}{\otimes} A\text{-mod}^{\mathbb{G}_m} \simeq A\text{-mod} \underset{QC(S_A)}{\otimes} QC(S_A)$.

~~Suppose~~ (I'll prove later). - back page

Suppose now $A\text{-mod} \curvearrowright C$ some cat. Then $\mathrm{QCoh}(S_A) \curvearrowright C$ as well. Let $Y \subset \mathrm{Spec}(A)$, conical, complement g[...]

then as showed before, $C_Y \underset{SG2}{\simeq} C \underset{Q(S_A)}{\otimes} Q(S_A)_{Y/\mathbb{G}_m}$.

Proof: ~~A~~ By de-equv. argument suffices to check $(A\text{-mod}_Y)^{\mathbb{G}_m}$ matches w/ $Q(S_A)_Y/\mathbb{G}_m$

under $(A\text{-mod})^{\mathbb{G}_m} \simeq Q(S_A)$.

| Note: need to check $A\text{-mod}_Y^{\mathbb{G}_m}$ makes sense.
Y conical ~~...~~ is wrt coh grading.

$A\text{-mod}_Y$ is invariant $\iff$ Y conical wrt. extra grading.

But A^{shift} classical means it's same grading!

So what's $(A\text{-mod}_Y)^{\mathbb{G}_m}$? It's graded A-modules whose support is on Y. ~~So~~ The rest is formal.

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$$\begin{array}{ccc} \mathcal{Y}[-1] & \to & pt \\ \downarrow & & \downarrow \\ pt & \to & \mathcal{Y} \end{array}$$

$\mathcal{Y}$ affine smooth.
~~classical~~.

Pg 3: $\mathcal{Y}[-1]$

Observation: A. it's quasi-smooth. (by definition)
B. it is non~~canonical~~ canonically isomorphic to

$\mathrm{Spec\,Sym}\, V^*[1]$, where $V = T_{pt}\mathcal{Y}$. (Worry not, $\mathcal{Y}$ is classical!)

B: suffices to restrict to formal neighborhood.
Non-canonically, this is $\mathbb{A}^n$ (exponential of V).
This now boils down to a pushout in ComAlg,
for which it's straightforward.

Pg. 6 ~~Prop: $Y = \mathrm{Spec}A$, $\mathrm{Spec}A = Y$ is $\ldots \Rightarrow T_Y$ compactly generated~~
~~Namely, $\mathrm{Cone}(t \xrightarrow{a} t)$ generate.~~

~~Annul support in means support of ... as graded A-mod.~~

Prop: $\left\{\begin{array}{l} Y \text{ arbitrary} \\ \{t_a\} \text{ generators, compact.} \end{array}\right. \Rightarrow \forall t.\ \mathrm{supp}\, t \subset T \Leftrightarrow \mathrm{supp}\, \mathrm{Hom}^\bullet_T(t_\alpha, t) \subset Y\ \forall\alpha$ as A-mod.

(Recall, $A \to \bigoplus_{n \geq 0} \mathrm{Hom}(t, t[2n])$ so it acts on $\mathrm{Hom}^\bullet(t_\alpha, t)$ by composition. (obj. in derived cat))

Pf:
Let a be $\in A$, homogeneous of deg $2k$.

$\mathrm{supp}(t) \subset Y_a \Leftrightarrow \mathrm{colim}(t \xrightarrow{a} t[2k] \xrightarrow{a} \ldots) = 0$.
$\Leftrightarrow \mathrm{Hom}(t_\alpha[m], \mathrm{colim}) = 0 \quad \forall \alpha, m$.
$\Leftrightarrow \mathrm{Colim}(\mathrm{Hom}(t_\alpha[m], -)) = 0 \quad \forall \alpha, m$.
$\Leftrightarrow \mathrm{supp}\, \mathrm{Hom}^\bullet_T(t_\alpha, t) \subset Y_a \quad \forall \alpha$.

Now observe $IC(pt \times_{\mathcal{Y}} pt)$ has one compact gen: $\Delta^{IC}_*(k)$.
Then Claim is trivial.