ON QUASI-SMOOTHNESS

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1. Introduction

We will define the notion of quasi-smoothness and develop its basic properties, following [AG].

2. Some recollections on the cotangent complex

All our schemes will be locally Noetherian and locally almost of finite type.

Classically, to a map of classical schemes $f : X \to Y$ we attach its *sheaf of relative differentials* $\Omega_{X/Y} \in \mathrm{QCoh}(X)$. Let's recall some of its formal properties: for a sequence of maps $X \xrightarrow{f} Y \to Z$, we have an exact sequence

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0,$$

and we have exactness on the left if f is smooth. Similarly, for a closed subscheme Y of X, we have that

$$\mathcal{C}_{Y/X} \to i^*\Omega_{X/\mathrm{pt}} \to \Omega_{Y/\mathrm{pt}} \to 0$$

is exact (where $\mathcal{C}_{Y/X}$ is the conormal sheaf of Y in X), which is also left exact for smooth Y. As often happens with four-term exact sequences that are five-term exact sequences under certain conditions, what's really happening is that the sheaf of relative differentials is the zeroth cohomology of a more fundamental object, the *cotangent complex*, and these exact sequences are the tails of long exact sequences arising from cofiber sequences of cotangent complexes.

We will not properly define the cotangent complex here, but instead summarize some of its basic properties.

To a map of derived schemes $f : X \to Y$ we associate the cotangent complex $T^*(X/Y) \in \mathrm{QCoh}^{\leq 0}(X)$. We have that $H^0(T^*(X/Y))^{\mathrm{cl}} \simeq \Omega_{X^{\mathrm{cl}}/Y^{\mathrm{cl}}}$, where the superscript cl on schemes denotes the underlying classical scheme and on sheaves on a scheme Z denotes their pullback to Z^{cl} along $Z^{\mathrm{cl}} \to Z$.

For a sequence of maps $X \xrightarrow{f} Y \to Z$, we have a cofiber sequence in $\mathrm{QCoh}(X)$:

$$f^*T^*(Y/Z) \to T^*(X/Z) \to T^*(X/Y).$$

Recall the notion of a smooth morphism of derived schemes: $X \to Y$ is *smooth* if $Y^{\mathrm{cl}} \times_Y X$ is classical, and the resulting map of classical schemes $Y^{\mathrm{cl}} \times_Y X \to Y^{\mathrm{cl}}$ is a smooth map of classical schemes in the usual sense. The cotangent complex allows for a convenient equivalent condition:

Theorem. *A map of derived schemes $X \to Y$ is smooth iff $T^*(X/Y)$ is a vector bundle on X, i.e. is Zariski-locally isomorphic to $\mathcal{O}_X^{\oplus n}$.*

We will not prove this theorem here; it is Proposition 2.2.2.4 in [TV]. We also have the following theorem, which follows from [Lur, Cor. 25.3.6.6]:

Theorem. *A map of derived schemes $X \to Y$ that induces an isomorphism $X^{\mathrm{cl}} \to Y^{\mathrm{cl}}$ is an isomorphism iff its cotangent complex $T^*(X/Y)$ vanishes.*

Finally, we will crucially use the following claim, which follows from [Lur, Prop. 2.7.3.2]:

Theorem. *Let X be a derived scheme, $M \in \mathrm{QCoh}(X)$ be a complex of $\mathcal{O}_X$-modules and M^{cl} be its pullback to X^{cl}. Assume further that M is perfect. Then:*

(1) M is n-connective iff M^{cl} is n-connective;

(2) M has Tor-amplitude in $[a, b]$ iff M^{cl} has Tor-amplitude in $[a, b]$.

3. Quasi-smoothness

From now on we will omit the word "DG", as everything will be derived by default.

Definition. A map of schemes $X \to Y$ is called *quasi-smooth* if the relative cotangent complex $T^*(X/Y) \in \mathrm{QCoh}(X)$ is perfect and of Tor-amplitude $[-1, 0]$.

Note that being perfect of Tor-amplitude $[a, b]$ is equivalent to locally being represented by a complex of free $\mathcal{O}_X$-modules of finite rank in degrees $[a, b]$ and zeroes in other degrees.

In what follows, we will give a convenient local characterization of quasi-smooth maps. First, let's see that it suffices to concentrate on quasi-smooth closed embeddings:

Lemma 3.0.1. *Any quasi-smooth map $X \to Y$ can be, Zariski-locally on X, factored as a composition of a quasi-smooth closed embedding, followed by a smooth map.*

Proof. Since we're working Zariski-locally on X, we may assume that both X and Y are affine and of finite type. This implies that for large enough n, we may factor $f : X \to Y$ as

$$X \xrightarrow{g} Y \times \mathbb{A}^n \xrightarrow{\pi} Y,$$

where g is now a closed embedding. Let's see that if f is quasi-smooth, then g is quasi-smooth. The maps above induce the following cofiber sequence in $\mathrm{QCoh}(X)$:

$$g^*(T^*(Y \times \mathbb{A}^n/Y) \to T^*(X/Y) \to T^*(X/(Y \times \mathbb{A}^n)).$$

If we denote by π' the unique map $Y \times \mathbb{A}^n \to \mathrm{pt}$, we have that

$$T^*(Y \times \mathbb{A}^n/Y) \simeq (\pi')^*T^*(\mathbb{A}^n/\mathrm{pt}) \simeq (\pi')^*\mathcal{O}_{\mathbb{A}}^{\oplus n}[0] \simeq \mathcal{O}_{Y \times \mathbb{A}^n}^{\oplus n}[0],$$

which further implies that $g^*T^*(Y \times \mathbb{A}^n/Y) \simeq \mathcal{O}_X^{\oplus n}$. Thus, we have the cofiber sequence

$$\mathcal{O}_X^{\oplus n} \to T^*(X/Y) \to T^*(X/(Y \times \mathbb{A}^n)).$$

Now, since the first term is perfect of Tor-amplitude $[0, 0]$ and the second term is perfect of Tor-amplitude $[-1, 0]$, the third term must be perfect of Tor-amplitude $[-1, 0]$. $\square$

Lemma 3.0.2. *A map of DG schemes $i : X \to Y$ is a quasi-smooth closed embedding if and only if Zariski-locally on Y, there exists a Cartesian diagram (in the category of DG schemes)*

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow & & \downarrow f \\ \{0\} & \longrightarrow & \mathbb{A}^n \end{array}$$

Proof. This is the proof found in [Khan], which really is the same as the proof in [AG]. Note that since $X \to Y$ is a closed embedding, $H^0(T^*(X/Y)) \simeq \Omega_{X^{\mathrm{cl}}/Y^{\mathrm{cl}}} \simeq 0$.

First, let's see that for a closed embedding $f : X \to Y$, $T^*[X/Y]$ being perfect of Tor-amplitude $[-1, 0]$ is equivalent to $T^*[X/Y][-1]$ being a vector bundle. The latter implying the former is immediate; for the converse, recall that $T^*(X/Y)$ being perfect of Tor-amplitude $[-1, 0]$ is equivalent to $T^*[X/Y]$ being locally represented by $\mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus m}$, placed in degrees -1 and 0 respectively. It suffices to check that $T^*[X/Y][-1]$ is a vector bundle after pulling it back to X^{cl}; there, it it locally represented by $\mathcal{O}_{X^{\mathrm{cl}}}^{\oplus n} \to \mathcal{O}_{X^{\mathrm{cl}}}^{\oplus m}$, placed in degrees 0 and 1, and we're given that H^1 of this complex vanishes. This implies that the complex is quasi-isomorphic to the kernel of $\mathcal{O}_{X^{\mathrm{cl}}}^{\oplus n} \to \mathcal{O}_{X^{\mathrm{cl}}}^{\oplus m}$, placed in 0-th degree, which is known to be a vector bundle classically, thus implying that $T^*[X/Y][-1]$ is a vector bundle, as required.

Now let's return to the lemma. The "if" direction is straightforward: consider the maps $\mathrm{pt} \xrightarrow{i} \mathbb{A}^n \to \mathrm{pt}$. They induce a cofiber sequence $i^*(T^*(\mathbb{A}^n/\,\mathrm{pt})) \to 0 \to T^*(\mathrm{pt}\,/\mathbb{A}^n)$ in Vect_k. Since $\mathbb{A}^n$ is smooth, its cotangent complex is quasi-isomorphic to $\Omega_{\mathbb{A}^n/\,\mathrm{pt}}[0] \simeq \mathcal{O}_{\mathbb{A}^n}^{\oplus n}[0]$, hence $T^*(\mathrm{pt}\,/\mathbb{A}^n) \simeq k^{\oplus n}[-1]$. Since the cotangent complex satisfies arbitrary base change, we have that $T^*(X/Y)$ is the pullback to X of $T^*(\mathrm{pt}\,/\mathbb{A}^n)$, i.e. $T^*(X/Y) \simeq \mathcal{O}_X^{\oplus n}[-1]$, which is a vector bundle.

Let's prove the converse. Since we're working Zariski-locally, we may assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine. Let F denote the homotopy fiber of $A \to B$ in $A-\mathrm{mod}$. Then, by [Lur, Cor. 25.3.6.1], we have that $H^0(F \otimes_A B) \simeq H^{-1}(T^*(X/Z))$. We won't prove this here, but will point out that this makes sense classically: recall that for classical schemes and the closed embedding $i : \operatorname{Spec} A/I \to \operatorname{Spec} A$, we have an exact sequence

$$I/I^2 \to i^*\Omega_{\operatorname{Spec} A/\,\mathrm{pt}} \to \Omega_{\operatorname{Spec}(A/I)/\,\mathrm{pt}} \to 0.$$

Classically, we have $F = I$ and $F \otimes_A B \simeq I/I^2$, and under this isomorphism, this exact sequence matches up with the long exact sequence arising from the cofiber sequence of cotangent complexes

$$i^*T^*(\operatorname{Spec} A/\,\mathrm{pt}) \to T^*(\operatorname{Spec}(A/I)/\,\mathrm{pt}) \to T^*(\operatorname{Spec}(A/I)/\operatorname{Spec} A)$$

(recall that $H^0(T^*(X/Y)) \simeq \Omega_{X/Y}$). The analogy isn't meant to be precise (though it probably can be made such via carefully taking H^0 of our objects).

Let $df_1, df_2, \dots, df_n$ be a basis for $T^*(X/Y)[-1]$. Using the isomorphism $H^{-1}(T^*(X/Y)) \simeq H^0(F \otimes_A B)$ and the fact that the map $H^0(A) \to H^0(B)$ is surjective (by definition of closed embeddings of DG schemes), we may lift $df_1, \dots, df_n$ to $H^0(F)$, and further lift those to F. Finally, since F is the fiber of $A \to B$, by taking their images $f_1, \dots, f_n$ in A we obtain a map $g : \operatorname{Spec} B \to pt \underset{\mathbb{A}^n}{\times} \operatorname{Spec} A$. This map induces an isomorphism on the underlying classical schemes, hence to show it's an isomorphism it suffices to show that the relative cotangent complex vanishes. Since the cotangent complex satisfies base change, we have that $T^*(\mathrm{pt} \underset{\mathbb{A}^n}{\times} Y/\mathrm{pt}) \simeq \mathcal{O}^{\oplus n}_{\mathrm{pt} \underset{\mathbb{A}^n}{\times} Y}[-1]$. The cofiber sequence arising from the maps $X \to \mathrm{pt} \underset{\mathbb{A}^n}{\times} Y \to \mathrm{pt}$ reads

$$g^*(T^*(\mathrm{pt} \underset{\mathbb{A}^n}{\times} Y/\mathrm{pt}) \to T^*(X/\mathrm{pt}) \to T^*(X/\mathrm{pt} \underset{\mathbb{A}^n}{\times} Y).$$

It remains to see that $g^*(T^*(\mathrm{pt} \underset{\mathbb{A}^n}{\times} Y/\mathrm{pt}) \to T^*(X/\mathrm{pt})$ is a quasi-isomorphism, which is true by construction: $g^*(T^*(\mathrm{pt} \underset{\mathbb{A}^n}{\times} Y/\mathrm{pt}) \simeq \mathcal{O}_X^{\oplus n}[-1]$, and the df_i precisely give an isomorphism

$$\mathcal{O}_X^{\oplus n}[-1] \simeq T^*(X/Y)[-1].$$

□

Thus, any quasi-smooth map $X \to Y$ can be Zariski-locally on X decomposed as $X \to \mathbb{A}^n \times Y \xrightarrow{\pi} Y$, where the map $X \to \mathbb{A}^n \times Y$ is obtained via derived base change from $\mathrm{pt} \to \mathbb{A}^m$. In particular, X is quasi-smooth if and only if it is locally given by a derived fiber product $\mathrm{pt} \times_{\mathbb{A}^m} \mathbb{A}^n$.

The following lemma shows that quasi-smoothness deserves to be called "the derived version of locally complete intersection".

Lemma 3.0.3. *A classical scheme X is quasi-smooth (when considered as a derived scheme) if and only if it is a locally complete intersection.*

Proof. We will be using the local description of quasi-smoothness provided by the previous lemma. Since both properties are local, the claim amounts to the following: the derived tensor product $k \otimes_{k[x_1,\ldots,x_n]} k[y_1, \ldots, y_m]$ has no higher cohomology (i.e. turns out to be classical) iff the sequence $(f_1, \ldots, f_n), f_i \in k[y_1, \ldots, y_m]$ that defines the map $k[x_1, \ldots, x_n] \to k[y_1, \ldots, y_m]$ is a regular sequence.

To show this, we will consider the Koszul complex of k as a $k[x_1, \ldots, x_n]$-module, which gives an explicit free (and hence flat) resolution of k. In fact, we will recall Koszul complexes more generally: for a commutative ring R and a sequence of elements $(r_1, \ldots, r_n), r_i \in R$, let $s : R^{\oplus n} \to R$ denote the natural map defined by $(r_1, \ldots, r_n)$ (given by $(a_1, \ldots, a_n) \mapsto \sum_i a_i r_i$). Denote by $K(s)$ the following chain complex of R-modules:

$$0 \to \wedge^n R^{\oplus n} \xrightarrow{d_n} \wedge^{n-1} R^{\oplus n} \to \ldots \to \wedge^1 R^{\oplus n} \simeq R^{\oplus n} \xrightarrow{(r_1,\ldots,r_n)} R \to 0,$$

where $d_k : \wedge^k R^{\oplus n} \to \wedge^{k-1} R^{\oplus n}$ is given by

$$d_k(a_1 \wedge \ldots \wedge a_k) = \sum_{i=0}^{k} (-1)^{i+1} s(a_i) \cdot a_1 \wedge \ldots \wedge \widehat{a_i} \wedge \ldots \wedge a_k.$$

For an R-module M, denote by $K(s; M)$ the tensor product of $K(s)$ with M. Recall that for an R-module M, $(r_1, \ldots, r_n)$ is called a regular sequence on M if r_k is a non-zerodivisor on $M/(x_1, \ldots, x_{k-1})M$ for every $1 \le k \le n-1$.

Lemma. *If $(x_1, \ldots, x_n)$ is a regular sequence on M, then $H_i(K(s; M) = 0)$ for all $i \ge 1$.*

Proof. We shall argue by induction on n. For $n = 1$, the Koszul complex is given by $M \xrightarrow{\cdot x_1} M$, hence $H_1(K(x_1; M)) = \mathrm{Ann}_M(x_1) = 0$ by definition.

For the induction step, let $s : R^n \to R$ denote the map given by $(x_1, \ldots, x_n)$, and $t : R \to R$ the map given by x_{n+1}. Observe that

$$\wedge^k(R^n \oplus R) \simeq \oplus_{i=0}^k \wedge^i (R^{\oplus n}) \otimes_R \wedge^{k-i} R \simeq \wedge^k R^{\oplus n} \oplus \wedge^{k-1} R^{\oplus n},$$

which gives rise to an exact sequence of chain complexes

$$0 \to K(s; M) \to K(s \oplus t; M) \to K(s; M)[-1] \to 0.$$

By considering the long exact sequence on homology, we get that $H_i(K(s \oplus t; M)) = 0$ for all $i \geq 2$. For $i = 1$, the long exact sequence tells us that

$$0 \to H_1(s \oplus t; M) \to H_0(s; M) \to H_0(s; M)$$

is exact. Unwinding the definitions, one finds that the connecting homomorphism $H_i(s; M) \to H_i(s; M)$ is always given by multiplication by x_{n+1}; in particular, this holds for $i = 0$. Since $H_0(s; M) \simeq M/(x_1, \ldots, x_n)M$, by definition of regularity we have that this map is injective, which shows that $H_1(s \oplus t; M) = 0$, as required. □

Lemma. *If R is a Noetherian local ring, M is a finitely generated R-module and $(x_1, \ldots, x_n)$ are non-units in R, then the converse of the previous lemma holds: $H_i(K(s; M)) = 0$ for all $i \geq 1$ implies $(x_1, \ldots, x_n)$ is regular on M.*

Proof. We shall argue by induction on n; the $n = 1$ case is clear, as the Koszul complex is given by $M \xrightarrow{\cdot x_1} M$. As before, let $s : R^n \to R$ denote the map defined by $(x_1, \ldots, x_n)$, and let $t : R \to R$ be the map defined by x_{n+1}. Recall the exact sequence $0 \to K(s; M) \to K(s \oplus t; M) \to K(s; M)[-1] \to 0$.

The long exact sequence in homology tells us that

$$H_i(K(s; M)) \xrightarrow{\cdot x_{n+1}} H_i(K(s; M)) \to 0$$

is exact for every $i \geq 1$; by Nakayama's lemma, this implies that $H_i(K(s; M)) = 0$ for every $i \geq 1$, which by induction implies that $(x_1, \ldots, x_n)$ is regular on M. Finally, by the long exact sequence we have that

$$0 \to H_0(K(s; M)) \xrightarrow{\cdot x_{n+1}} H_0(K(s; M))$$

is exact. Since $H_0(K(s; M)) \simeq M/(x_1, \ldots, x_n)M$, we precisely have that $(x_1, \ldots, x_{n+1})$ is regular, as required. □

Finally, the proof of the original claim is as follows:

Classical and classical l.c.i imply that locally, $X \simeq k \otimes_{k[x_1,\ldots,x_n]} k[y_1, \ldots, y_m]$ where the map $k[x_1, \ldots, x_n] \to k[y_1, \ldots, y_m]$ is given by a regular sequence. By the first theorem above, this implies that when we take a free resolution of k as a $k[x_1, \ldots, x_n]$-module and tensor with $k[y_1, \ldots, y_m]$, the result has no higher cohomology, i.e. the classical tensor product coincides with the derived one, which implies that X is quasi-smooth.

Classical and quasi-smooth imply that locally, $X \simeq k \otimes_{k[x_1,\ldots,x_n]} k[y_1, \ldots, y_m]$ and is classical. This implies that the stalks of X are classical l.c.i's, by the second theorem above. Since X is a finite type k-algebra, it follows (see [00SH]) that X is l.c.i., as required.

□

4. The scheme of singularities and the singular codifferential

4.1. **The scheme of singularities.** Let X be a quasi-smooth scheme.

Definition. The *tangent complex* $T(X)$ of X is defined to be the dual of $T^*(X)$.

Note that since X is quasi-smooth, $T^*(X)$ is perfect, hence it is a reasonable operation to consider its "naive" dual. We also observe that since $T^*(X)$ has Tor-amplitude $[-1,0]$, $T(X)$ has Tor-amplitude $[0,1]$; in fact, these conditions are equivalent.

We shall now define a classical scheme $\mathrm{Sing}(X)$ to measure how far from being smooth X is.

Definition. For X a quasi-smooth scheme, we define $\mathrm{Sing}(X) := (\mathrm{Spec}_X(\mathrm{Sym}_{\mathcal{O}_X}(T(X)[1])))^{\mathrm{cl}}$.

Note that since we're taking the underlying classical scheme, we have

$$\mathrm{Sing}(X) = \mathrm{Spec}_{X^{\mathrm{cl}}}(\mathrm{Sym}(\mathcal{O}_{X^{\mathrm{cl}}}(H^1(T(Z)))),$$

where $H^1(T(Z))$ is considered a sheaf on X^{cl} (i.e. we take its pullback along $X^{\mathrm{cl}} \to X$). Also note that since $T^*(Y)$ is perfect, pullback and taking dual commute: $f^*(T(X)) \simeq (f^*(T^*(X)))^\vee$.

We will also need a relative version of this.

Definition. Let X, Y be quasi-smooth schemes, and $f : X \to Y$ be any map. Define $\mathrm{Sing}(Y)_X = (\mathrm{Spec}_X(\mathrm{Sym}(\mathcal{O}_X)(f^*(T(Y))[1])))^{\mathrm{cl}}$.

As before, we have that $\mathrm{Sing}_Y(X) = \mathrm{Spec}_{X^{\mathrm{cl}}}(\mathrm{Sym}_{\mathcal{O}_{X^{\mathrm{cl}}}}(H^1(f^*(T(Y))))$. A map $f : X \to Y$ induces a map $T(X) \to f^*(T(Y))$ and hence a map (the *singular codifferential*) $\mathrm{Sing}(f) : \mathrm{Sing}(Y)_X \to \mathrm{Sing}(X)$. This allows for a convenient reinterpretation of quasi-smooth maps between quasi-smooth schemes.

Theorem. *Let X, Y be quasi-smooth, and $f : X \to Y$ be a map. Then f is quasi-smooth iff the singular codifferential $\mathrm{Sing}(f) : \mathrm{Sing}(Y)_X \to \mathrm{Sing}(X)$ is a closed embedding.*

Proof. In this proof, all sheaves are considered to be pulled back to X^{cl}; we will omit the notation. Note that $\mathrm{Sing}(f)$ being a closed embedding is equivalent to $H^1(T(X)) \to H^1(f^*(T(Y)))$ being surjective.

f being quasi-smooth, i.e. $T^*(X/Y)$ being perfect of Tor-amplitude $[-1,0]$, is equivalent to $T(X/Y) := (T^*(X/Y))^\vee$ being perfect of Tor-amplitude $[0,1]$. The latter fits into a cofiber sequence

$$T(X/Y) \to T(X) \to f^*(T(Y)).$$

Since $T(X)$ and $f^*(T(Y))$ are perfect of Tor-amplitude $[0,1]$, the cofiber sequence implies that $T(X/Y)$ is perfect of Tor-amplitude $[0,2]$. Now, $H^2(T(X/Y)) = 0$ is equivalent (by the long exact sequence in cohomology) to $H^1(T(X)) \to H^1(f^*(T(Y)))$ being surjective. We've previously brought an argument that having Tor-amplitude $[0,1]$ and having H^1 zero is in fact equivalent to being a vector bundle. An identical argument shows that being perfect of Tor-amplitude $[0,2]$ and having H^2 zero implies being perfect of Tor-amplitude $[0,1]$: indeed, Tor-amplitude $[0,2]$ is equivalent to locally being represented by

$$\mathcal{O}_{X^{\mathrm{cl}}}^{\oplus n} \to \mathcal{O}_{X^{\mathrm{cl}}}^{\oplus m} \xrightarrow{f} \mathcal{O}_{X^{\mathrm{cl}}}^{\oplus l},$$

where f is surjective. This is tautologically quasi-isomorphic to

$$\mathcal{O}_{X^{\mathrm{cl}}} \to \ker f,$$

and $\ker f$ is indeed a vector bundle by virtue of being the kernel of a map of vector bundles.

□

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