# Dmod(Gr) part 2. (David)

Thm from last time:

$$\Gamma : D_\kappa(Gr) \longrightarrow \hat{g}_\kappa\text{-mod}.$$

exact & faithful.

($\kappa$ critical is the hard part)

Summary of last time:

Sketch of proof:

factor this as

$$D_\kappa(Gr) \xrightarrow{\pi^!} "D(G(\mathcal{K}))" \xrightarrow{\Gamma} \left(\hat{g}_\kappa \times \hat{g}_{2\kappa_{crit}-\kappa}\text{-mod}\right)^{G(O)_{dR}}$$

taking R-inv vectors

$$M \longmapsto \mathrm{Hom}_{\hat{g}_R}(V_\kappa, M) \qquad \downarrow$$

$$\left(\hat{g}_\kappa\text{-mod}\right)^{G(O)_{dR}}$$

$$V_\kappa = \mathrm{Ind}_{g(O)+\mathbb{C}}^{\hat{g}_\kappa} \mathbb{C}_{triv.}$$

$V_{2\kappa_{crit}-\kappa}$ is projective in $\hat{g}_{2\kappa_{crit}-\kappa}\text{-mod}^{G(O)}$ if $\kappa$ negative.

everything should be exact (last Hom from projectivity).

"$D(G(K))^{G(O)}$" := modules $M$ over the CADO $\mathcal{D}_{K,x}(G)$ supported at $x \in X$ where the induced right action of $\hat{g}(O) \oplus \mathbb{C}$ on $M_x$ integrates

$$A_{\hat{g}_k} \to \mathcal{D}_{K,x}(G) \leftarrow A_{\hat{g}_{2k_{crit}-k}}$$

chiral universal enveloping algebra

# Dirty & Quick Understanding of Chiral Algebras.

Commutative $D_X$-algebra $\rightsquigarrow$ commutative algebra

Lie$^*$-algebra $\rightsquigarrow$ Lie algebra

Chiral algebra $\rightsquigarrow$ associative algebra

In $A_{g,k}$ case there's the following exact seq:

$$0 \longrightarrow dR(D_x, A_{g,k}) \longrightarrow dR(D_x^*, A_{g,k}) \longrightarrow A_{g,k\,x} \longrightarrow 0$$

zeroth deRham Coh

disk

(shriek fiber at $x$)

translating to

$$0 \longrightarrow I \longrightarrow U(\hat{g}_k)/\underline{1} = 1 \longrightarrow V_k \longrightarrow 0$$

$$\mathfrak{Z}(A_{g,k}) \stackrel{\text{def}}{=}: \mathfrak{Z}_g$$

Commutative D-algebra $\mathrm{Fun}(O_p)$.

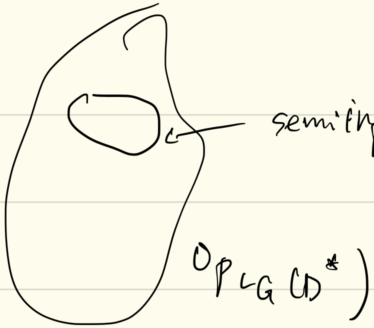
In other words, [Chiral modules over $A_{g,k}$]

$[\ ] \cong \hat{U}(\hat{g})$-mod

have a $\mathrm{Fun}(O_{p_{Lg}}(D^*))$-action.

The action on $V_{\mathrm{crit}}$ factors thru

$$\mathrm{Fun}(O_{p_{Lg}}(D))$$




So $V_{crit}$ can't be proj. (There'll be a fix.)

$\hat{\mathfrak{g}}_{crit}\text{-mod}_{reg}$ ($\hat{\mathfrak{g}}_{crit}\text{-mod}_{\widetilde{reg}}$) are modules scheme theoretically (set theoretically) supported on $Op_{^LG}(D)$.

Claim: $V_{crit}$ is a proj. generator of $\hat{\mathfrak{g}}_{crit}\text{-mod}_{reg}$.

remark: it generates $\hat{\mathfrak{g}}_{crit}\text{-mod}_{\widetilde{reg}}$ but not proj.

Back

$$D_{crit}(Gr) \xrightarrow[\text{exact (provable)}]{} (D_{\mathfrak{g}_{crit}}\text{-mod})^{G(O)} \xrightarrow[\text{exact (obvious)}]{} \hat{\mathfrak{g}}_{crit}\text{-mod}^{G(O)}$$

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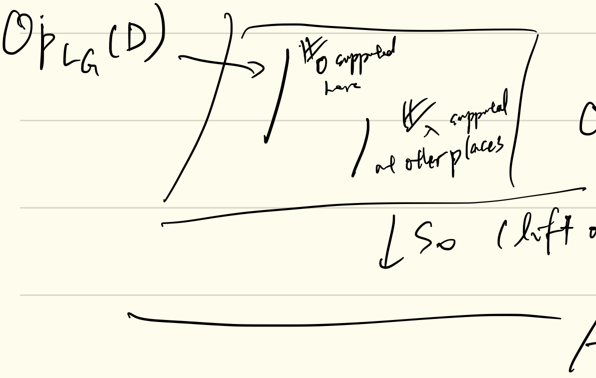
$\xrightarrow{①} \hat{g}_{crit}\text{-mod}^{G(O)}_{reg} \xrightarrow{②} \hat{g}_{crit}\text{-mod}_{reg}.$

① exact cause it's both left and right adjoint (see below)

$M \in \hat{g}_{crit}\text{-mod}^{G(O)}$

has canonical decomposition $M_{reg} \oplus M_{non\text{-}reg}$




$\downarrow S_0$ (lift of Casimir) (Segal Sugawara element)

$A^1$ $\quad S_0(\text{supp}\,\mathbb{V}_\lambda) = $ Casimir of $\lambda$.

$\mathbb{V}_\lambda = \mathrm{Ind}_{g(O)\oplus\mathbb{C}}^{\hat{g}}(V_\lambda)$ generates $\hat{g}_{crit}\text{-mod}^{G(O)}$.

② isn't exact. Here comes universal renormalized algebra.

$$\begin{tikzcd}
D_{g,crit}\text{-mod}^{G(O)} \arrow[r] \arrow[dr, hook] & \hat{g}_{crit}\text{-mod}^{G(O)} \arrow[r, "①"] & \widetilde{op} \arrow[r, "②"] & \sim \\
 & \hat{g}^{ren,d}_{crit}\text{-mod}^{G(O)} \arrow[u] \arrow[r] & \hat{g}^{ren,d}_{crit}\text{-mod}^{G(O)}_{\widetilde{reg}} \arrow[d, "②'"] & \\
 & & \hat{g}^{b,d}_{crit}\text{-mod} \arrow[uur, bend right] &
\end{tikzcd}$$

"d" doubled

all vertical maps are exact (they are forgetful)

all bottom horizontal arrows are exact.

This solves the problem.

By exactness above, just need to show

②' exact.

$\hat{\mathfrak{g}}^b_{crit}$ -mod, dechiralized

Call a $\hat{U}(\hat{\mathfrak{g}}_{crit})$-mod <u>central</u>

if $I = \ker\left(\mathrm{Fun}(Op_{^LG}(D^\times)) \to \mathrm{Fun}(Op_{^LG}(D))\right)$

acts trivially on it.

eqv: module being central as a chiral module.

motivation: imagine having a family of

$\hat{U}(\hat{\mathfrak{g}}_{crit})$-mod $M_\hbar$ w/ $M_0$ central.

Then can get a $\hat{\mathfrak{g}}^b_{crit}$-action on $M_0$.

$$0 \to U(\hat{\mathfrak{g}}_{crit})/I \to \hat{\mathfrak{g}}^b_{crit} \to \Omega^1(Op_{^LG}(D)) \to 0$$

also "$U(\hat{\mathfrak{g}}_{crit})$ acts on $M_0$"

comes from "$I/\hbar$ acting on $M_0$"

and $Op_{^LG}(D)$ is Poisson structure.

We need to know the commutators (some lift)

$$[i_1/h, i_2/h] = \{\tilde{i_1}, \tilde{i_2}\}/h.$$

$$\cancel{\mathrm{Fun}(Op_{{}^LG}(D)) \otimes U(g)/I \oplus I/h}$$

all necessary commutators

$$\overset{?}{\cong} \hat{g}^{b}_{crit}.$$

$$\Omega^1_{Op_{{}^LG}(D)} \xhookrightarrow[\text{cokernel}]{\text{finite dim projective}} T_{Op_{{}^LG}(D)} Op_{{}^LG}(D).$$

$\hat{g}^{ren,d}_{crit}$ is for situations where you have two $\hat{g}_{crit}$-actions so that the $\mathrm{Fun}(Op_{{}^LG}(D^\times))$ actions are identified.

(finite dim situation: Lie algebra embeds to diff op. in two ways but they agree on center)

note: $$U\hat{g}_{crit} \xrightarrow{L} D_{g,K} \xhookleftarrow{R} U\hat{g}_{crit}.$$

send the centers to the same thing.

Pf: centralizer $(\mathrm{im}\, L) = \mathrm{im}\, R$

$\sim\!\sim$ $(\mathrm{im}\, R) = \mathrm{im}\, L$.

So we get (iso is not identity).

$$\hat{g}^{ren,d} \longrightarrow D_{g,crit}.$$

this gives

$$\begin{tikzcd}
D_{g,crit}\text{-mod} \arrow[r] & \hat{g}^{ren,d}\text{-mod} \arrow[d] & \\
& \hat{g}^{ren,d}\text{-mod}_{reg} \arrow[r, "\cong"] \arrow[d] & \hat{g}^{b,d}\text{-mod} \\
& \tilde{g}^{ren,d}\text{-mod}_{\hat{reg},\xi} \arrow[ur] &
\end{tikzcd}$$

(spirit: Kashiwara thm)

differ by $\Omega^1$-algebroid.