DAG Seminar, 10/11.

Review of $(\infty,1)$-Cat and all that.

Given the audience I'll not motivate ... okay maybe one example.

$$\begin{tikzcd} X_1 \arrow[r, "g_X"] \arrow[d, "f_1"'] & X_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "g_Y"] & Y_2 \end{tikzcd}$$

Base change: $g_Y^* f_{2*} \to f_{1*} g_X^*$ (Derived).

False for usual schemes.

($X_i = \operatorname{Spec} A_i$, $Y_i = \operatorname{Spec} B_i$, get $B_1 \overset{L}{\otimes}_{B_2} A_2 \to A_1$).

Lemme just remind you what ∞-Cat are:

Δ ~~simplices~~ numbers. $\Delta^n = \mathrm{Hom}_{\Delta^{op}}[\cdot, [n]]$ simplices.

Set_Δ simplicial set: $\Delta^{op} \to Set$.

$\Lambda_i^n = \Delta^n$ minus ith face. $(0 \le i \le n)$.

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d, hook] & S \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

all $\exists$: Kan complex = space.

$0<i<n$: weak complex = ∞-cat.

(Explain why it's a category).

Impose Kan model structure on Set_Δ.

($|X| \simeq |Y|$, Kan fibration, $X_n \hookrightarrow Y_n$).

~~right lifting~~ right lifting against $\Lambda_i^n \subseteq \Delta^n$

All obj cofib.

fib = Kan complexes.

$Spc = N(Set_\Delta^\circ)$ ← fib-cofib. nerve

1-Cat = Similar (Joyal cat structure)

Basic Dictionary:

Functor: morphism of sSet.

Fully faithful: obvious.

1-fully-faithfull: Hom → Hom mono.

1-full subcat: take sub-Hom-set, needs to contain all isos are close under composition.

w/ Cartesian Fibrations.

Def. $F: D \to C$ functor.

$\alpha: d_0 \to d_1 \in D$. α <u>Cartesian</u> (over C) $\Longleftrightarrow$ $\forall d' \in D$.

$$\mathrm{Map}_D(d', d_0) \to \mathrm{Map}(d', d_1) \underset{\mathrm{Map}(F(d'), F(d_1))}{\times} \mathrm{Map}(F(d'), F(d_0)). \text{ iso.}$$

those are spaces

(~~[illegible]~~ ~~[illegible]~~ a def that's not black box (requires working w/ sSet). I don't really want to go in there ...).

Prop. (Lurie Remark 2.4.1.4). f Cartesian iff $\forall n \geq 2$, lift exists.

$$\begin{tikzcd}
\Delta^{\{n-1,n\}} \arrow[d, hook] \arrow[drr, "f"] & & \\
\Lambda^n_n \arrow[rr] \arrow[d, hook] & & D \arrow[d, "F"] \\
\Delta^n \arrow[rr] \arrow[urr, dashed] & & C
\end{tikzcd}$$

Def. $F: D \to C$ is <u>Cartesian fibration</u> if.

$\forall\, c_0 \to c_1 \in C$, $d_1 \in D$, $F(d_1) \simeq c_1$.

$\exists\, d_0 \to d_1$ Cartesian.

$$\begin{tikzcd}
F(d_0) \arrow[r] \arrow[d, "\simeq"] & F(d_1) \arrow[d, "\simeq"] \\
c_0 \arrow[r] & c_1
\end{tikzcd}.$$

(This is the same thing as Cartesian in fibred category).

Def. <u>Strict transform</u>: Functors (over base space) that send Cartesian arrows to Cartesian arrows.

<u>Over and Under Cats</u>

First define join of sSet: $(X * Y)_n = X_n \cup Y_n \cup \bigcup_{i+j=n-1} X_i \times Y_j$.

$Y \in$ sSet, $p: Y \to C$.

$$C_{/p} := \mathrm{Hom}(X, C_{/p}) \simeq \mathrm{Hom}_p(X * Y, C).$$

(skipping actual definition).

$$C_{p/} := \mathrm{Hom}(X, C_{p/}) \simeq \mathrm{Hom}_p(Y * X, C).$$

Special case $Y = \Delta^0$ $p: \Delta^0 \to c \in C$. write C/c and $C_{c/}$.

$1\text{-}Cat/_C$:= obvious.

$coCart/_C$:= full subcat, maps are coCart.

$(coCart/c)_{strict}$:= 1-full-subcat

Thm.

$$(coCart/c)_{strict} \simeq Funct(C, 1\text{-}Cat).$$

$$(Cart/c)_{strict} \simeq Funct(C^{op}, 1\text{-}Cat).$$

Remark: - this is compatible w/ inclusion $Spc \to 1\text{-}Cat$.

i.e. $(0\text{-}coCart/c)_{strict} \simeq Funct(C, Spc)$.

F need to be coCartesian $D \to C$ s.t. $\forall c \in C$. $D \times_C \{c\} \in Spc$.

- this is the ∞-version of the following Grothendieck construction:

C cat $\quad X: C^{op} \to SmallCat$.

$\Downarrow$

$\tilde{C}/c$: obj = $(c \in C, \eta \in X(c))$.

mor: $(c, \eta) \xrightarrow{f, \alpha} (c', \eta')$

$f \in C$, $\alpha \in X(c)$.

$\alpha: \eta \to X(f)(\eta')$.

This allows some cool stuff.

Yoneda.

$$\begin{tikzcd} Fun(\Delta^1, C) \arrow[r, "(ev_0, ev_1)"] \arrow[dr, "ev_0"'] & C \times C \arrow[d, "pr_1"] \\ & C \end{tikzcd}$$

this is a strict transform, so we get from straightening $\forall$ a natural transformation of $C^{op} \to 1\text{-}Cat$:

$(c \mapsto C_{c/}) \rightsquigarrow (c \mapsto C)$.

This natural transformation is an obj in:

$\mathrm{Fun}(\Delta^1, \mathrm{Fun}(C^{op}, 1\text{-Cat})) \cong \mathrm{Fun}(C^{op}, \mathrm{Fun}(\Delta^1, \mathrm{Cat}))$.

~~where~~ given by $c \mapsto (\underbrace{C_{c/} \to C}_{\text{this is in } 0\text{-coCart}/C})$.

So straightening gives: $C^{op} \to \mathrm{Funct}(C, \mathrm{Spc}) \leftarrow$ Yoneda.

Adjoint

$F : C_0 \to C_1$ functor.

$\wr\|$

$\Delta^1 \to 1\text{-Cat}$.

$\downarrow$ unstr.

$\check{C} \to \Delta^1$ for some $\check{C}$. This is coCart.

If it is also Cart, then say F has right adjoint.

Then apply str. get $\Delta^{1\,op} \to 1\text{-Cat}$.

$\downarrow$

$C_1 \to C_0$. the right adjoint functor.

This can be generalized to produce. $C_I : I \to 1\text{-Cat}$

$\Downarrow$

$C_{I^{op}}^R : I^{op} \to 1\text{-Cat}$.

Partial-adjoint

$\check{C} \to \Delta^1$ is only coCart.

take $C_1' \subset C_1$ full subcat:

$\{c_1 \mid \exists c_0 .\ c_0 \to c_1 \in \check{C}\}$

$\downarrow$ Cartesian.

Δ^1

$\check{C}' \subset \check{C}$: $\check{C}'_0 = C_0$, $\check{C}'_1 = C'_1$.

Then $\check{C}' \to \Delta^1$ is by definition Cartesian.

$\Rightarrow$ get $C'_1 \to C_0$. The <u>partial right adj.</u>

Kan Extensions

$F : D \to C$ functor. E ∞-cat.

$$\mathrm{Funct}(C, E) \longrightarrow \mathrm{Funct}(D, E).$$

(L/R) partial adj =: (L/R) Kan extension.

In particular, take $C = *$

$$LKE = \operatorname*{colim}_{D} : \mathrm{Funct}(D, E) \to E.$$

$$RKE = \lim.$$

If $\Phi : D \to E$. $\forall c$. $\operatorname*{colim}_{D \times_C C_{/c}} \Phi$ $\exists$. Then

so does $LKE_F(\Phi)$, given by $LKE_F(\Phi)(c) =$

Here's a construction which Gaitsgory claims to be important. I don't see why yet, would love comments.

<u>Def</u>. 1-full subcat of 1-Cat. called presentable ∞-cat.

1-Cat_{prs}. Obj: $\underbrace{\text{presentable}}_{\text{accessible + containing small limits}}$ ∞-Cat

Mor: Colimit-preserving functors.

Thm.

$L : 1\text{-}Cat_{prs} \to 1\text{-}Cat$.

a) $f \in 1\text{-}Cat_{prs} \Rightarrow L_* f$ has right adjoint.

b) $G : D \to C$ G accessible. ~~the~~ then G has left adjoint in $1\text{-}Cat_{prs}$.
$\cap$ preserve limit
$1\text{-}Cat_{prs}$

Now fix $C_I : I \to 1\text{-}Cat_{prs}$.

(Thm) get $C^R_{I^{op}} : I^{op} \to 1\text{-}Cat$.

attach a final object $*$ to I, get I'.

extend C_I to $C_{I'} : I' \to 1\text{-}Cat_{prs}$

w/ $* \longrightarrow \operatorname{colim} C_I$.

(Thm) get $C^R_{I'^{op}} : I'^{op} \to 1\text{-}Cat$.

restrict to I^{op}: is $C^R_{I^{op}}$.

This gives functor $\operatorname{colim} C_I \longrightarrow \lim C^R_{I^{op}}$ (*)

(Lurie, 5.5.3.4) This is an equivalence.

Define. $C_i \xrightarrow{ins_i} C_*$ in obvious way.

$ev_i : \lim_{I^{op}} C^R_{I^{op}} \to C_i$.

Equivalently, ins_i is the left adjoint of ev_i (upon $C_* \simeq \lim C^R_{I^{op}}$).

(*) is then given by universal property.